

Mathemat. Vol 2.

A

Preliminary Discourse

T O

An intended TREATISE

ON THE

FLUXIONARY METHOD.

Largely explaining

The Nature and Peculiarity

O F

T H A T D O C T R I N E,

In a familiar and easy Manner.

By JOHN ROWNING, M. A. *K*

Author of the *Compendious System of Natural Philosophy.*

Ex his Principiis via sternitur ad majora.

Newt. Quad. Curv.

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TO

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THE A T D O C T R I N E

As a method and early manner

BY JOHN ROWLING, M.A.

Author of the Grammar of the English Language

For his Principles of Grammar and Syntax

LONDON

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TO
THE REVEREND
DOCTOR NICOLLS,
VICAR OF CRIPPLEGATE,
BY WHOSE INSTRUCTION
IN THE
UNIVERSITY OF CAMBRIDGE,
THESE STUDIES WERE BEGUN
AND CARRIED ON
TO A CONSIDERABLE LENGTH,
THIS
PRELIMINARY DISCOURSE,
AS A PUBLIC TESTIMONY
OF RESPECT,
IS MOST HUMBLY PRESENTED
BY
HIS MOST DEVOTED
AND MOST HUMBLE SERVANT,
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Preliminary DISCOURSE

On the Nature and Application of the Fluxionary Method: With a short Account of a Controversy occasioned by the Manner in which that Method was at first explained, and the Rules of it demonstrated by Sir Isaac Newton, and other Mathematicians at Home and Abroad.

I. **T**HE great Improvements which have been made by the Mathematicians of this and the last Century, both in the higher Geometry, and in Natural Philosophy, is in a great measure owing to the *Method of Fluxions*. What rendered this Method instrumental to those Improvements, was, that *curvilinear* Figures are treated by the Assistance hereof, almost with the same Facility and Readiness, as if they were *right-lined*; which, as every one the least acquainted with Geometry, cannot but be sensible, is a mighty Advantage.

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2 *A preliminary Discourse.*

In Consequence of this Advantage, a large Field for Discoveries and Improvements in Natural Philosophy readily opened itself, the Quantities considered in that Science, *viz.* Powers, and their Effects, being representable by geometrical Figures, and in most Cases by such only as are curvilinear; and the Properties of the Figures, by which Quantities are capable of being represented, always point out to us the Affections and Relations of the Quantities themselves: Of which I purpose to give some Instances in this Discourse, to shew, how it is, that Geometry is so exceeding useful, and even necessary, in the Business of Natural Philosophy. It was by the Help of this Method of Fluxions, together with another Method, called, *The Doctrine of prime and ultimate Ratios*, that Sir *Isaac Newton* was enabled to make such wonderful Discoveries in the Nature and Constitution of our System; and to demonstrate the Truth of the Propositions in which he has comprized them: His Demonstrations of which are much greater Objects of Admiration, than the Discoveries themselves. Hence it is, that those *Views* of his Discoveries, we have been more than *once* entertained with under different Titles, appear languid and flat, *viz.* because his Proofs, which are the Spirit and Life of his Performances, being unintelligible to common Readers, are, for that Reason, in such Treatises, always left out of the Account. But to the Point —

II. The

II. The *Doctrine of Fluxions* is a peculiar Kind of *Analysis*, or Method of Investigation, or Resolution; wherein Quantities are not considered, as of a permanent or stated Magnitude, agreeably to the Methods made Use of in other Computations, but are conceived to be in a perpetual or uninterrupted *Flux*; that is, in a State of *continual Variation* from one Degree of Magnitude to another. They are supposed gradually *to increase* or *to decrease*, so far as the Case may require, and this under certain and determinate Regulations; and thence the Relations they bear to each other, and their Properties are discovered.

III. The Principle this Method is founded upon, so far as relates to the finding the Relation or Proportion Quantities bear to each other, is this, *viz. That all Quantities are greater or less, according to the Time and Degree of Velocity or Quickness with which they were, or might have been, produced*, that is, might have arrived from Nothing to their present Magnitude, *by a regular and gradual Increase*: And consequently, that when two or more Quantities are, or might have been, produced by increasing with equal Degrees of Velocity in equal Times, those Quantities are equal to each other; and that when the Velocities they shall be thus produced with, shall all the time of the Increase bear any

4. *A preliminary Discourse.*

given and determinate Proportion to each other, the Quantities themselves shall be, the one to the other, in that Proportion. This I shall explain by the following easy Instances.

Let AB and CD, in Figure 1. be two circular Archs, whose common Center is O, and which are terminated by the two right Lines OA and OB; and let it be proposed to discover the Relation those two Archs bear to each other in Point of Magnitude; that is, to find how many times one of them is longer than the other, their Radii OA and OC being given.

To perform this by the common Geometry, the Way would be to compare those Archs with the whole Circumferences ABFA and CDHC; and then by comparing those Circumferences with each other, it might be discovered, that those Archs are to each other, as their Radii OA and OC. But the Fluxionist proceeds quite otherwise, thus: He would consider with what comparative Velocities those Archs might have been produced, had they arose from mathematical Points at the same time, and increased together till they became, the one AB, the other CD. In order to this, he would possibly have supposed a Line, as OM, to have moved from the Situation OA to OB, turning upon the Center O with any Velocity made Choice of at Pleasure (it matters not how fast or how slow) and

A preliminary Discourse. 5

and that the Archs arose out of the Points A and C the Moment the Line OM began to move from OA; and that, when that Line came to OM, the Arch AB was arrived to the Length AM, and the other to the Length CP; and that, when the regulating Line OM came to ON, the former Arch became AN, the latter CQ; and so on, till the former was arrived to the Length AB, the latter to CD. He would then discover, by Rules we have for such Purposes, that the Archs lengthening under this Regulation, must have increased all the Way with Velocities, Rates or Paces proportionable to their Distances from the Center O; and would from thence infer, that those Archs would at all times, and therefore also, when they are compleated, be to each other, as those Distances; that is, as their Radii OA and OC. Which is the same Conclusion that might have been deduced from common Geometry; but, as is evident, is here obtained on a very different Principle, and by quite another Method of reasoning.

Or, it may be supposed, that the Line OM moves the contrary Way, *viz.* from OB to OA, causing the Archs to *decrease*, as it moves along, while they vanish, the one at A the other at C. In this Case, it is clear, the Archs will decrease perpetually with the same Velocities the Points M and P shall be carried with by the Motion of the Line OM; and it

6 *A preliminary Discourse.*

would appear by the Rules of this Method hinted at above, that those Points will be carried all the Way with Velocities proportionable to their respective Distances from the Center O ; the Archs therefore will decrease all the Way with Velocities proportionable to those Distances; and since they will both of them necessarily lose their whole Lengths, or be reduced to Nothing, in the same time by so decreasing, it is manifest, that before the Decrease began, they must have been to each other in that Proportion.

If the Relation the two Areas ABO and CDO bear to each other, were required, the Line OM might be supposed to turn upon the Center O , as before, from OA to OB , and the said Areas to arise, the former out of the Line OA , and the latter out of the Line OC , the Instant OM sets out from OA ; and to increase together in such manner, that when the Line OM is come from OA to ON , the Areas shall be ANO and CQO ; and that, when OM is come to OB , the Areas shall have arrived to their full Magnitude ABO and CDO ; then shall these Areas be found, by the Rules of this Method, to have increased all the Way with Velocities proportionable to the Squares of the Lines OA and OC ; from whence it is to be inferred, that they are to each other, as those Squares; which is the well known Proportion such Areas bear to each other. Take another Instance.

Let

A preliminary Discourse. 7

Let ACB , in Fig. 2. represent a Semi-circle, whose Diameter is AB ; and let AKB represent a Semi-ellipse, whose longer Axis is AB , and shorter one is twice OK . Any where upon the Line AB , erect the Perpendiculars PM and QR : and, as they generally express it, let the two Areas of those Figures be described by the Motion of one of those Perpendiculars, as PM , moving parallel to itself from A to B : That is, let that Perpendicular be supposed to move upon the Diameter AB from A to B , continuing perpendicular thereto all the Way, and lengthening or shortening as it moves along, as the curve Line $AMRCB$ requires; and let the two Areas, the Moment PM sets off at A , be supposed to arise out of that Point, and when PM is come to QR , let the Areas be supposed to have arrived, the one to the Magnitude ARQ , the other to the Magnitude AIQ ; and when PM is come to OC , let the Areas be, the one ACO , the other AKO ; and so on, still regulated in their Increase by the Motion of that Perpendicular PM , till they are compleated, and become, the one ACB , and the other AKB . In which Case it will be found, from the known Properties of these Figures, and by the Rules of this Method, that the semicircular Area ACB will have increased all the Way, so many times faster than the semi-elliptical one AKB , as the Line OC is longer than OK ; that is, as twice OC , or AB the

8 *A preliminary Discourse.*

Diameter of the Circle, is longer than twice OK the shorter Axis of the Ellipse: A manifest Indication, that those Areas are to each other in that Proportion: Which is the Relation such Areas are otherwise demonstrated, by the Writers on conic Sections, to bear to each other.

Thus the comparative Magnitudes of geometrical Quantities may, in some Cases, be immediately deduced from the comparative Rates, Paces, or Degrees of Velocity they shall be found to increase with; the finding which Rates of Increase is, as I shall shew in some Instances afterwards, beyond all Imagination easy: And then, when the comparative Magnitudes of the Quantities are thus found, if one of them be already known, or be such that it may be easily measured, the rest, as is evident, may readily be known from thence.

IV. I have not mentioned the foregoing Cases, as Instances of the fluxionary Method, but only to explain and illustrate the above-mentioned Principle on which I observed that Method is founded, and as it were to initiate the Reader therein by giving him an Insight how it is, that in some Cases the comparative Magnitudes of Quantities may immediately be deduced from the comparative Rates they shall increase with. For it very frequently, nay almost always happens, that whatever Regulation you suppose the Quantities to increase under,

under, they shall not be found to increase, the one exactly the same Number of Times faster than the other all the Way, or from first to last; in which Case, the Proportion they shall bear to each other cannot be immediately deduced from the comparative Rates they shall increase with. For Instance: Suppose that while two Quantities are increasing together, it should appear from their respective Properties, and the Regulation we suppose them to increase under (which are the Things we always reason from); I say, suppose it should appear, that at first setting off, one of them should increase ten times faster than the other, by and by eleven times faster, and at last twelve times as fast. Here it might indeed be inferred, that one of those Quantities is more than ten times larger than the other; because it increases upon the whole more than so many times faster; and it may be inferred, that it is less than twelve times as large; but to say from thence precisely how many times it is larger, is not so easy. How the Magnitude of Quantities is to be found in this Case is now to be explained.

V. To give a clear and adequate Conception of this, it is requisite previously to consider, what it is that in mathematical Language is meant by finding the Magnitude of a Quantity. By the finding the Magnitude of a Quantity then is meant either,

1. The

10 *A preliminary Discourse.*

1. The finding a *numerical Expression*, as 100, or 1000, or some other Number: When we have found an Expression of that sort for the Quantity sought, we say, the *Magnitude* of that Quantity is found, and inquire no farther. Or,

2. The finding an *algebraical Expression*, as ax , bxy , $\sqrt{ax+xx}$, or the like: Here also, when an Expression of this Sort is found for the Quantity, its *Magnitude* is said to be ascertained; nothing more being then requisite, than to substitute Figures for the Letters of which that Expression consists; or, as we commonly term it, *to put it into Numbers*.

3. If you ask a Geometrician how large the Quantity is, he will tell you perhaps, that he has found it equal to the Area of a Circle whose Diameter is so or so; to an Arch perhaps of such a Circle, whose Tangent, or whose Sine perhaps, is so or so; to the Content perhaps of some other Figure, which he will tell you is to be constructed, that is, to be drawn, after this or that manner; and the like. Here indeed the *Magnitude* of the Quantity cannot, strictly speaking, be said to be discovered; because a Computation, perhaps of no easy Kind, may still be requisite for the finding how great that Area, that Arch, or that Content may be. However, the having obtained an Expression of this Kind for the Quantity sought, is also, among Mathematicians,

A preliminary Discourse. **II**

thematicians, considered as having found the Value or Magnitude thereof. But observe, every Quantity cannot be expressed any one of these Ways: Happy are we sometimes, when we can find an Expression for it from them all combined together.

VI. It being usual, in the fluxionary Method, to endeavour to find an Expression of the second Kind, *viz.* an algebraical one, for the Quantity sought, when such Expression can be had; I begin with Cases of that Sort.

To explain this, it is proper to take notice, that, if one of the Letters or Members in any algebraical Expression, as in axx , $b-x$, or the like, should increase or be taken larger and larger, while the other Letters or Members therein remain as they are, or, more properly speaking, while they continue to denote Quantities of unalterable Value, the Expression itself would necessarily *increase*, or *decrease*. Thus, if x in the first of those above-mentioned, *viz.* in axx , should *increase*, the Expression itself would also *increase*; because the larger x is, the larger will axx be, provided the other Letter a remains as it is: But if x in the second Expression, *viz.* in $b-x$ should increase, that Expression would decrease; because x in that, being a negative Quantity, the larger it is, so much less is the Quantity $b-x$. Farther, if you suppose x in the algebraical Expression $ax-xx$ to increase continually, and a to be constant or unchangeable,
you

12 *A preliminary Discourse.*

you will find, that the Expression itself shall grow larger and larger for a while, *viz.* till x is become equal to half a , and after that it shall grow less and less. In short, algebraical Expressions may be found, that shall increase or decrease according to very different Laws of Variation, while one of the Letters, Members, or Terms contained therein shall continue to increase with an uniform or even Pace. This being observed,

VII. Let $ARTB$, in Fig. 3. represent the whole, or any Part of a geometrical Figure, whose Nature, that is, some one at least of whose general Properties, is already known; and let it be proposed to find an algebraical Expression, that shall denote or express the Magnitude of its Area ATB , or any Part thereof, as AQR or AST , &c. In order to this, let the said Area be supposed to be described, produced or generated, (as they term it) by a Line as PM , making any Angle with the Line AB , and moving parallel to itself with any Velocity taken at Pleasure, from A to B , in the same manner as was mentioned for the two Areas, in Fig. 2. *viz.* so, that if QR and ST be drawn parallel to PM , when P comes to Q , PM shall come to QR , and the Area shall be AQR ; and when P comes to S , PM shall come to ST , and be equal to it, lengthening or shortening, as it moves along, as the Curvature of the Figure requires, and the Area shall be AST ; and so on, till
 PM

PM comes to the Point B, and vanishes there, and the Area be compleated. Then an algebraical Expression is to be sought for, of such Sort, that if one Term, Letter or Member therein should increase at the same Rate with the Line AP, the whole Expression would increase at the same Rate with the Area APM; for then taking that Letter or Member in that Expression equal to AP, the Expression itself, (except in some particular Cases to be mentioned by and by) will be equal to the Area APM; and taking that Letter equal to the whole Line AB, the Expression, will be equal to the whole Area ATB.

VIII. To shew the Reason of this, it is farther to be remarked, that although an algebraical Expression may observe various Laws of increasing at different Times, that is, may increase first slowly, and then faster, or the contrary, while one of its Terms shall continue to increase at an even Pace, as above-mentioned; yet the Rate it shall itself increase at, at those different Times, may be expressed the same Way, or by one and the same general Expression, or Theorem. Thus, the Rate the first of the above-mentioned algebraical Expressions, axx , shall increase at, while x increases at the Rate Unity, shall, as will appear by the Rules of this Method to be delivered in the Book itself, be expressible by $2ax$, or $2ax \times 1$; or, if the Letter x in that Expression be supposed to increase continually at

14 *A preliminary Discourse.*

at the Rate $\dot{x}$, or Dot x , (which, for Reasons I shall mention afterwards, is the usual Way of expressing it) that is, if we denominate the Rate we suppose the Quantity x in that Expression to increase at, by the Character Dot x , the Rate the Expression itself shall increase at, shall be expressible by $2ax\dot{x}$. This is called the Fluxion of the Quantity axx , and implies, that if x in that Quantity should continue to increase at the Rate Dot x , (meaning thereby any uniform or constant Rate greater or less) the Expression axx would necessarily and consequentially increase at the Rate $2ax\dot{x}$; that is to say, that when x in that Expression begins to increase, and is as yet nothing, that Expression shall then increase at the Rate $2a \times 0 \times \dot{x}$, or Nothing, that is, it shall not increase at all; when x in that Expression is by increasing arrived to the Magnitude Unity, the Expression axx shall then increase at the Rate $2a \times 1 \times \dot{x}$, or $2a\dot{x}$, that is, twice a times faster than we suppose the Term x to increase; and that when x is become 2, axx shall increase at the Rate $2a \times 2 \times \dot{x}$, or 4 times a times faster than x ; and so on. In like manner, by the same Rules $a - 2xx\dot{x}$ shall be found to be the Fluxion of the Quantity $ax - xx$, and implies, that if x in that Quantity, or Expression, $ax - xx$, should increase at the Rate $\dot{x}$, the Expression itself would in Consequence thereof increase at a Rate always expressible by $a - 2xx\dot{x}$, that is,

that

A preliminary Discourse. 15

that when x begins to increase and is as yet nothing, $ax - xx$ shall increase at a Rate expressible by $a - 2 \times 0 \times \dot{x}$, or $a\dot{x}$, that is a times faster than we suppose x to increase; and when x is become One, and is increasing on at the same Rate $\dot{x}$, as before, $ax - xx$ shall be at that Instant increasing at the Rate $a - 2 \times 1 \times \dot{x}$, and so on. In like manner,

The Fluxion of $x^{\frac{m}{n}}$, supposing x to increase at the Rate $\dot{x}$, shall be $\frac{m}{n} x^{\frac{m-n}{n}} \times \dot{x}$. And if a be any given Quantity, and x be supposed to increase at the Rate $\dot{x}$, as we have all along supposed, then

The Fluxion of $ax^{\frac{m}{n}}$ shall be $a \times \frac{m}{n} x^{\frac{m-n}{n}} \times \dot{x}$.

The Fluxion of $ax^{-\frac{m}{n}}$ shall be $a \times -\frac{m}{n} x^{-\frac{m+n}{n}} \times \dot{x}$.

The Fluxion of $\frac{2x\sqrt{ax}}{3}$ shall be $\sqrt{ax} \times \dot{x}$.

The Fluxion of $\sqrt{a+x}$ shall be $\frac{\dot{x}}{2\sqrt{a+x}}$.

&c.

The Method of finding these Fluxions will be largely explained in the Book itself.

IX. Ob-

IX. Observe farther, that when a geometrical Figure is generated or described in such manner as was mentioned for the Figure ARTB, in Fig. 3. and when the Lines PM, QR and ST are all perpendicular to the Base, which we will at present suppose, the Way to find the Rate the Area shall increase at, at any Place or Time, suppose for Instance when it is APM, is to multiply PM by the Rate you suppose AP to lengthen at; or, which is the same Thing, by the Rate you suppose the Line PM itself to move forwards with. If therefore you denominate AP the Distance of the Perpendicular PM from the Point A by some general algebraic Character, as x suppose, and, from the Nature and Properties of the Figure, (which, as I said before, we always suppose to be known) and from AP (or x), you compute the Value of PM, by which means you will obtain a general algebraic Theorem for that Perpendicular at whatever Distance you take it from A, because the Term x , being a general algebraic Character, may denote any Distance greater or less; and then, if you multiply that Value by the Rate you suppose AP to lengthen at, suppose by $\dot{x}$, you have a general algebraic Expression for the Rate the Area APM shall increase at all along, however various it may be, while the Base AP increases all the Way at the uniform or even Pace $\dot{x}$. That Expression will be what is called the *Fluxion* of the Area, and if

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it comes out the same, suppose, with any of the Fluxions mentioned in the foregoing Article, it is an Indication, that (except, as I observed before, in some Cases to be mentioned by and by) the algebraical Expression, which has that Fluxion, is such, that if x , the flowing Letter, Member or Term therein should increase at the same Rate with the Base AP, the Expression itself would increase at the same with the Area; and therefore, as observed above, that, if you take the increasing Term x in that Expression equal to any Length of the Base AP, the Expression itself shall be equal to, or shall denote the Area APM corresponding to that Base; and consequently that, if you take that Letter equal to AB, the Expression shall then be equal to, or shall denote the Magnitude of the whole Area ATB.

X. The finding such algebraical Expression is called *finding the Fluent*. That such Fluent or algebraical Expression will in some Cases be equal to the Quantity sought, will appear from the following Instance.

Let ABC, in Fig. 4. be a Parabola; call the Parameter of it a ; let AB be its Axis, and let PM and QR be perpendicular thereto; and let the Area of it be supposed to be described by the Perpendicular PM moving parallel to itself from A towards BC: Call the indeterminate Distance AP, x ; then, as is demonstrated by all the Writers on this Figure,

18 *A preliminary Discourse.*

gure, the Perpendicular PM will be equal to, or expreffible by, $\sqrt{ax}$, and is therefore by them already calculated to our Hands. Call the Velocity you fuppofe the Line PM to move with in defcribing the Figure, $\dot{x}$; then multiply the Value of that Perpendicular, *viz.* the Quantity $\sqrt{ax}$ by $\dot{x}$, and you have $\sqrt{ax} \times \dot{x}$, which, for the Reasons above-mentioned, is a general Exprefſion denoting the Rate the Area APM ſhall increaſe at, at all Magnitudes thereof: It implies, that when the Baſe AP , or x , begins to increaſe, and is as yet a Point in A , the Area ſhall then increaſe not at all, the Quantity $\sqrt{ax} \times \dot{x}$ being then $\sqrt{a0} \times \dot{x}$, or Nothing; and that when AP , or x , is one, the Area APM correſponding thereto, ſhall increaſe at the Rate $\sqrt{a1} \times \dot{x}$, or $\sqrt{a} \times \dot{x}$, and ſo on. But it was obſerved above, that the algebraical Exprefſion $\frac{2x\sqrt{ax}}{3}$

is a Quantity whoſe Fluxion is alſo $\sqrt{ax} \times \dot{x}$, conſequently it is an Exprefſion that when x therein ſhall increaſe at the ſame Rate with AP the Baſe of the Parabola, ſhall itſelf (ſuppoſing a here the ſame alſo with a there) increaſe at the ſame with the Area correſponding to that Baſe: And that it ſhall be equal to that Area may be thus demonſtrated.

When the deſcribing Line PM ſets off at A , and the Baſe AP , or x , is as yet Nothing, the

the Area A P M is also Nothing, as appears by Inspection of the Figure ; and likewise, when x in the algebraical Expression $\frac{2x\sqrt{ax}}{3}$ begins

to increase, and is therefore as yet equal to

Nothing, the Expression itself is $\frac{2 \times 0 \sqrt{a0}}{3}$, or

Nothing. That Expression therefore and the Area are not unequal when the Increase begins ; since then that Expression increases constantly at the same Rate with the Area, while the Term x therein increases at the same Rate with the Base, it is clear, that when that Term x is equal to the Base, the Expression itself will be equal to the Area, I mean the Area that insists upon, and is described in the same Time with that Base ; and therefore, by taking x in that Expression equal to any Length of the Base A P, we render the Expression itself equal to the Area corresponding thereto. For, by taking x equal to A P, we take the Times of Increase equal ; but when the Times of Increase are equal, the Quantity

$\frac{2x\sqrt{ax}}{3}$ and the Area, which constantly increase along with each other, must be equal also.

XI. This is a Circumstance always to be attended to ; for it by no means follows, that because two Quantities increase with equal Paces, they shall themselves be equal ; for, as is manifest, unless they be equal at first setting

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off, or when the Increase begins, their increasing with equal Paces will be so far from making them equal, that it will continue them as unequal as they were at first. It is only the *cotemporary* Fluents, as they express it, that shall be equal to each other; that is, it is only what Quantities shall have gained by increasing with equal Paces during equal Times, and not what they themselves shall be at the End of those Times, that shall be equal to one another. Suppose, for Instance, that the Figure ABC had been of such sort, that its

Fluxion would have been found $\frac{x}{2\sqrt{a+x}}$;

this, as observed above, (§ 8.) is the Fluxion of the algebraical Expression $\sqrt{a+x}$, yet that Expression would not have been the true Value of the Area APM; for tho' it be a Quantity which, while the Term x therein increases at the same Rate with AP, or x , in the Figure, does itself increase at the same with the Area, yet it is always larger than that Area by the Quantity $\sqrt{a}$; for, when AP, or x , in the Figure, and also x in that Expression are Nothing, the Expression itself is then $\sqrt{a+0}$, or $\sqrt{a}$, but the Area is then Nothing; that Expression therefore exceeds the Area at that Time, and consequently all along, by that Quantity $\sqrt{a}$; to render it equal to it therefore it must be diminished by that Quantity, that is, that Quantity must be deducted

deducted from it; by which means it will be made $\sqrt{a+x} - \sqrt{a}$, which would be the true Value of the Area APM. The making this Alteration in the algebraical Expression is called, *Correcting the Fluent*. There are more Cases of it, which will be largely explained in the Book itself.

XII. And now may appear the Reason, why it is usual to denominate the Rate we suppose any Quantity, or Term, to increase at, by the same Letter, or Character, we denominate the Quantity itself (as the Rate we suppose x to increase at) by $\dot{x}$: it is, that they may refer to each other; by which means it appears at first Sight, which of the Letters in any fluxionary Expression, as, in $\sqrt{ax \times \dot{x}}$,

$\frac{\dot{x}}{2\sqrt{a+x}}$, and the like, is supposed to flow

or increase, and which not. In like manner, when any other Letter or Symbol, as y or z , &c. is taken to denote an increasing Quantity, the Fluxion of that Quantity, or the Rate we assume or suppose it to increase at, is usually expressed by the same Letter y or z , &c. but distinguished therefrom by a Dot or Tittle over it, as above-mentioned; and is to be read or pronounced Dot y , or Dot z , &c. This is the Notation the *English*, following Sir *Isaac Newton*, the great Author of this Method, generally make Use of: And hence it is, that in the Writings of the *English* upon

22 *A preliminary Discourse.*

this Subject those Characters $\dot{x}$, $\dot{y}$, &c. so oft occur. The Foreigners, instead of Dot x , Dot y , &c. use dx , dy , &c. the Reason of which shall be explained by and by.

But as from an algebraical Expression, as $\sqrt{a+x}$, $\frac{2x\sqrt{ax}}{3}$, &c. we can find its Fluxi-

on, or fluxionary Expression, $\frac{\dot{x}}{2\sqrt{a+x}}$,

$\sqrt{ax} \times \dot{x}$, &c. by certain Rules for that Purpose, so by the Converse of those Rules we can find the former from the latter, that is, the algebraical Expression from its Fluxion. When therefore we have found the Fluxion of the Quantity sought, as explained above, the next Step is, from that Fluxion to find the algebraical Expression whose Fluxion it also is, then that algebraical Expression, when duly corrected, if need be, which I here mention once for all, is to be taken for the Quantity sought.

Or, in determining the Areas A P M, in Fig. 3 or 4. an algebraical Expression might have been sought for such, that when one Term or Letter therein should have increased at the same Rate with the Line P M, or even with the curve Line A M, the Expression itself should have increased at the same with the Area, that algebraical Expression, or Fluent as they term it, provided we take that Term

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or Letter therein equal to PM , or AM , would have given the Area APM .

XIII. We have supposed the Areas in Fig. 3 and 4 to be generated or described by a Line, as PM , moving parallel to itself from A towards B . The same Areas might have been conceived, as described under a very different Regulation, thus: Imagine a Line, as PM , Fig. 5. to turn upon the Point P , as a Center, from PA to PM , from thence to PR , and so on, and to lengthen or shorten, as the Form of the Figure requires: And when the Line PM is come to PR , let the Area be APR ; and when PM is come to PT , let the Area be supposed to have arrived to the Magnitude APT , and so on. Then, let an algebraical Expression be sought, that shall increase at the same Rate with the Area, while some one Term, Letter or Member therein shall increase at the same with AM or PM , then taking that Letter or Term equal to AM or PM , the algebraical Expression shall be equal to the Area APM , at whatever Distance the Line PM is supposed to be from the Line PA ; that is, whatever be the Magnitude of the Angle APM .

If the curve Line AM , in Fig. 3 or 5. were required, let the Figure be supposed to increase under either of the forementioned Regulations; then from the Nature of the Figure, and by Rules to be mentioned in the Book itself, compute at what Rate AM shall

24 *A preliminary Discourse.*

lengthen, while AP or PM in Fig. 3. or while PM in Fig. 5. shall increase, (suppose at the Rate $\dot{x}$ or $\dot{y}$) then from the fluxionary Expression that shall arise, find an algebraical one which shall have the same fluxionary Expression also for its Fluxion; then taking the flowing Term in that Expression equal to any Length of AP or PM in Fig. 3. or of PM in Fig. 5. you have an Expression denoting the Value, that is, the Length of the curve Line AM .

If the Magnitude of a Solid, as ABC in Fig. 6. were required. Let its Axis be AO , and suppose the Solid to be described by a Plane, as MN , parallel to the Base BC , and moving from A towards O , with the Velocity $\dot{x}$; then find a general Expression, that shall denote the Rate the Solid shall increase at under these Circumstances; which is to be done, as shall be demonstrated afterwards, by computing the Value of that Plane at the indeterminate Distance AP from the Point A , and multiplying that Value, if the Plane be perpendicular to AO , by $\dot{x}$, viz. by the Rate you suppose the Plane to move with; then from the Expression that shall arise, find an algebraical one which shall have that Expression for its Fluxion, and taking the flowing Term in the algebraical Expression equal to AP , the algebraical Expression shall thereby be rendered equal to the Solid AMN ; if therefore you take that Term equal to the

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the whole Axis A O, you have the Value or Magnitude of the whole Solid.

XIV. But it shall frequently happen, that whatever Regulation you suppose a Quantity to increase under, you shall meet with a Fluxion of so perverse a Kind, that no algebraical Expression whatever, nor no finite Number of such, shall be, even capable of having that Fluxion, or of flowing at that Rate. This happens when the Quantity sought cannot be expressed by any finite Number of Terms whatever, which is very often the Case. Then the Difficulty begins to rise. Then it is your utmost Skill and Address may be required. You are then to look out for some geometrical Figure, of such Sort, that when some one Part thereof (some Line for Instance) shall increase at the same Rate with the Base, or other Dimension, of the Quantity sought, either the Arch or some other Part of that geometrical Figure shall increase at the same Rate with the Quantity sought; then, if you take that Part or Line in that Figure equal to the Base, or other Dimension of the unknown Quantity, the said Arch or other Part of that Figure shall be equal to the Quantity sought. In this Case you obtain a *geometrical Expression* for the Quantity sought. This is also called, *Finding the Fluent*. But it is here also to be understood, that this Fluent is, if need be, to be corrected,

26 *A preliminary Discourse.*

corrected, as before : Because otherwise, notwithstanding it is a Quantity which increases or flows at the same Rate with the Quantity sought, it may, or may not, be equal thereto.

In order to find such Figure or geometrical Quantity, you are first to have Recourse to the *Measures of Angles*, or to the *Measures of Ratios*; by the former of which is meant *Archs of Circles*, such being the Measures of their corresponding Angles at the Center; by the latter, *hyperbolic Areas* or *the Logarithms*, either of which are the Measures of Ratios : By this means perhaps you may find a Quantity which alone, or together with others, shall have the same Fluxion with your Quantity sought. This is what is looked upon, as the second Degree of Resolution : It falls short of the first, *viz.* of that we have all along been explaining, because neither the Archs of Circles, nor the Areas of the Hyperbola, can be expressed in algebraic Terms, and therefore they cannot be accurately measured, or expressed in Numbers. They have however been already computed to very great Exactness by several learned Men, and are extant, the former in the Tables of Sines and Tangents; the latter, in those of Logarithms. If this fails, your next Recourse is to the Archs of the Hyperbola or of the Ellipsis, or to both these combined together; which may require a most intimate Acquaintance with those
Curves,

Curves. If this fails, then to the Areas or Archs, &c. of some other mathematical Figure you can measure: In which Cases, the greatest Dexterity and algebraical Knowledge imaginable is often requisite in order to *transform* your fluxionary Expression, so that it may appear in the most commodious and proper Form. If all fail, your *derniere* Resort is to infinite Series, the lowest Degree of Resolution. In this Case the fluxionary Expression you have to deal with, is to be thrown into an *Infinite Series*, by Sir *Isaac Newton's* Binomial Theorem, or otherwise; (the Algebraists know how.) That Series will contain an infinite Number of fluxionary Expressions, for each of which you may readily find a Fluent expressed in algebraic Terms; then so many of those Fluents as you shall think proper, must be made Use of for the Value of the Quantity sought.

But, whereas these Things are sometimes Matter of great Difficulty, learned Mathematicians, and Men of great Skill in this Doctrine, particularly the late Mr. *Cotes*, Professor of Astronomy and Experimental Philosophy at *Cambridge*, in his *Harmonia Mensuratum*, Mr. *Emerson* in his Treatise of Fluxions [and others] have taken the Pains to calculate them ready to our Hands, and have composed Tables, which generally consist of two Columns; in the former of which you have great Variety of fluxionary Expressions, and
over

28 *A preliminary Discourse.*

over against them in the other, their respective Fluents. The Fluents that correspond to your Fluxion in those Tables, is, when duly corrected, to be taken for the Value, Magnitude or Measure of your Quantity sought.

XV. All Quantities, either *increasing* or *decreasing*, are, in the Language of the Fluxionists, which we shall have Occasion to adopt as we go along, stiled *variable* or *indeterminate Quantities*; and are said to *flow*, to be in a perpetual *Flux*; from whence this Doctrine, founded on that Consideration, has the Name of *Fluxions*. Such Quantities are also called *Fluents*, and sometimes *flowing Quantities*; and the Rates they vary with, whether by increasing or decreasing, are their *Fluxions*. On the other hand, such as are not considered as varying in Magnitude, are said to be *constant* or *invariable*. Those whose Magnitude is known are, as in common Geometry, called *given Quantities*.

Thus, in Fig. 3. the Lines AP, PM and AM, and the Area APM, all which vary in Magnitude in Consequence of the Motion of the describing Line PM; as likewise any algebraical Expression, as ax , byy , or the like, one or more of whose Terms or Members, as x , or y , is supposed to increase or decrease, are called *Fluents* or *flowing Quantities*; as are also the increasing or decreasing Terms x or y themselves; and the Rates, Paces or Degrees of Swiftneſs they increase or decrease with, are their

their respective *Fluxions*. Whereas Lines, or other Quantities, as AB, suppose in Fig. 3, 4, and 5. which are not at all affected by the Motion of the describing Line PM, are said to be *constant* or *invariable*.

XVI. This Doctrine is usually distinguished into two Parts; the former of which is called the *direct Method of Fluxions*; the latter, the *inverse Method*. Under the former is included whatever relates to the finding of a *Fluxion*, whether it be from the Properties of a Quantity as yet unknown, or from an algebraical Expression; and under the latter, the finding the *Fluent*, whether it be algebraical, geometrical, or mixed, from the *Fluxion*. The Rules for both which shall be laid down and demonstrated with the utmost Perspicuity and Plainness, in the Treatise for which this preliminary Discourse is intended.

XVII. I shall now proceed to set forth the same Things in a very different Light, viz. that in which the foreign Mathematicians generally represent them; and then, after having made some Observations relative to what shall have been delivered, shall consider some other Uses of this Method of an higher Kind.

XVIII. The foreign Mathematicians, most of whom learned this Method from *Leibnitz*, and to whom they contend the Honour of the Invention is due, found it upon a Principle
in

30 *A preliminary Discourse.*

in Appearance very different, viz. *That every Quantity consists of an infinite Number of Parts, and that the Quantity itself is equal to the Sum total of those Parts.* Accordingly, I have observed, some of the *Germans* have called that Part of the Method, which relates to the finding the Magnitude of a Quantity, *Summatio* and *Integratio*, as being a summing up of those little Parts to find the whole. This Principle, however true it may be, seems but a precarious Bottom, or Foundation; because one can form no Idea of a Quantity infinitely little: But the Point lies here; such infinitely little Quantity, as they conceive of it, may be *expressed*; and the Expression is all in all with a Mathematician. In order therefore to give a clear Idea of the Method taken in this View, I shall shew how it is they conceive of an infinitely small Quantity; how they express it; and how the Magnitude of Quantities may be discovered by them.

XIX. An infinitely small Quantity then, as they conceive of it, is not any random indeterminate Point, or uncertain Fragment of a Quantity, either physical or mathematical; but such Part only, as the Quantity, if it were to *increase* after any regular Manner, might *gain* thereby in each infinitely small Portion of Time; or, which is the same Thing, such Part as the Quantity might *lose* in such infinitely small Time, were it to *decrease*. For Example:

Let

A preliminary Discourse. 31

Let AB , the Base of the Figure ACB (Fig. 7.) be supposed to be divided into an infinite Number of infinitely short Lines, as AD , DE , EF , &c. and from the Extremity of each, let Lines, as DR , ES , &c. be supposed to be drawn parallel to each other: Let also the Figure or Area ACB be supposed to arise out of the Point A , and to increase continually, gaining in the first infinitely small Portion of Time, the Area of ADR ; in the next, the Area $DESR$; in the next, $EFTS$, and so on: Then will these Areas ADR , $DESR$, &c. be the infinitely little Parts of this Figure. Or, the Area of the same Figure may be supposed to arise out of the Line AP , see Fig. 8. and to increase continually, so as to gain in the first Portion of Time, the infinitely little Area APS ; in the next, SPT ; then TPV , &c. then will those infinitely little Spaces be the infinitely little Parts of the Figure. Observe therefore, that as, according to the former Conception of this Method, the same Figure, or Quantity, may be considered as increasing under different Regulations, as explained above; so may the same Figure have its infinitely little Parts very different. They respect the Manner in which the Figure is, or might be, supposed to increase, as well as the Form of it. The Figure is not supposed to be rent, as I observed above, into infinitely small Pieces in an arbitrary Manner; but divided into such regular and determinate Parts,

32 *A preliminary Discourse.*

Parts, as it may be supposed to be ever *gaining* while it increases, or *losing* while it decreases, under any fixed and stated Regulation. Hence it is, that, though no one has an adequate Idea of these infinitely little Quantities, yet the Mathematicians know precisely, and most determinately, both how to express, and how to treat them. Hence no Confusion ever arises from their Use.

XX. The Lines DE, EF, FG, &c. Fig. 7. being supposed infinitely small, the Lines DR and ES are, in Consequence of that Supposition, to be deemed precisely equal to each other; and therefore to find an Expression for the infinitely small Area DRSE, if DR, ES, &c. be all perpendicular to the Base, which we will at present suppose, multiply either DR, or ES, by DE, and the Product arising therefrom shall be the Expression for that Area. Again, EF being infinitely little, ES and FT are also precisely equal; for to be different they must be finitely distant from each other, which is not consistent with the Supposition that EF is infinitely short; and therefore either ES or FT multiplied by EF will be an Expression for the infinitely small Space ESTF, and so on. If therefore you suppose all these infinitely small Distances DE, EF, FG, &c. equal to one another, and denominate any one of them by some algebraic Character, as x ; and then, from the Nature of the Figure, raise a general Theorem that

that shall express any one of these Perpendiculars DR, or ES, &c. as was before directed for finding the general Value of PM, in Fig. 3 and 4. and then multiply that Theorem by $\dot{x}$, you have then a general Expression for any one of those infinitely little Parts DRSE, ESTF, &c. of which the Figure consists, or into which it is divided. But it is evident, that by this means, provided you make use of the same Characters for the Quantities concerned, you shall obtain the very same Expression, as when, in order to find the Fluxion of the Area, we multiplied the general Value of PM, in Fig. 3 or 4. by Dot x . The very same Expression therefore which, according to the one Conception of the Method, denotes the Fluxion of the Figure, in the other denotes the Magnitude of any one of those infinitely little Parts into which it is divided. So, in like manner, the Expression for any one of those infinitely little Spaces APS, SPT, &c. of which the Area ABC, in Fig. 8. is, according to this Conception, supposed to consist, will be found to be the very same, (provided the same Characters be made use of in the Computation) that the Fluxion of that Area would be, according to the other.

XXI. But how shall we be able to cast up the *Sum total* of all those infinitely little Quantities, to find the Magnitude of the Area they compose? The Way is thus: To find another

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Quantity

Quantity such, that when one of its Terms; Members or Parts shall be equal to the Base of the Figure, and consequently may be supposed to contain the same infinitely little Parts with that Base, shall itself contain the same infinitely little Parts with the Area; By which is meant no more, than finding such a Quantity, that its infinitely little Parts shall be expressible in the same Manner with those of the Area: for that Quantity shall be equal to the Area corresponding to that Base; and we shall then have the Sum total of all those infinitely little Parts of which the Area consists actually expressed in that Quantity. The finding such Quantity answers to finding the Fluent from the Fluxion, according to the other Conception of the Method; and is to be performed precisely in the same Manner. But it is always to be understood, that, if it shall happen, that such Quantity shall be found to contain any finite Parts over and above, or besides its infinitely small ones, more than the unknown Quantity does besides those infinitely small ones which appertain to it, then such finite Parts in both shall be done away, or at least rendered equal in both, by deducting the Overplus, or supplying the Deficiency. This answers to correcting the Fluent in the other Way, and is performed in the same Manner.

It appears therefore that, properly speaking, there is no *Summatio* or summing up in the Case:

Case : That is only a figurative Way of speaking, for the finding an Expression for a Quantity from the Expression for one of its infinitely little Parts. One^d might as soon count the Sands upon the Sea Shore, as cast up an infinite Series of Quantities, each of which is infinitely small, in the strict and literal Sense of the Word, *Summing up*.

XXII. But there is one Circumstance more, relative to these infinitely little Quantities, which it may not be amiss to take Notice of here, lest our Idea of them should be too narrow and contracted ; which is this, *viz.* that the Supposition that Quantities are infinitely small, does by no means imply any Equality among them. For, as in the other Conception, a Quantity may be supposed to increase faster or slower ; so in this, a Quantity may be considered as every Instant of Time gaining more or less : But what an increasing Quantity is gaining in each Instant of Time, are the very Things they mean by its infinitely small Parts ; there is therefore an absolute Necessity that such Quantities should be considered, as different one from another : They would otherwise be very incompetent Measures or Representatives of the *unequal* Rates Quantities may be supposed to increase at. I just mention this, that the Reader may not be surprised, when he finds them halved and quartered, multiplied and divided, and, in short, treated as subject to all the Rules both

36 *A preliminary Discourse.*

of Arithmetic and Geometry. How this comes to be consistent with their being infinitely small, I shall consider more particularly towards the End of this Discourse, where I shall endeavour to unravel some of those Difficulties, this Method has been charged with on Account of such Quantities.

XXIII. It would be needless to explain or open the Method any farther, as to this Conception of it, it being evidently the same with the other: For if instead of Dot z , Dot y , Dot x , &c. we put dz , dy , dx , &c. and use the same other Characters or Symbols a *Leibnitzian* would do, we shall fall into the very same Expressions; and Expressions are what we work with, and what we want. The Difference between them consists only in this, that in the one we proceed by the Rates, Velocities, or Degrees of Quickness wherewith Quantities increase; in the other, by what they are ever gaining thereby: Which two Things, being ever proportionable to each other, admit of one common Measure or Expression. Hence it is, that we naturally fall into the same Manner of *Expressing* in both; and hence it is, that both Methods do in Practice coincide and make one.

XXIV. These infinitely little Quantities are sometimes considered as *indefinitely small* only; sometimes as *less than any that can be given*; but what is meant by them under these, or the like palliating Appellations, is still the same

same in all Respects whatever. The Foreigners call them *Differences*, as being supposed to be the Difference between any Quantity, and the same Quantity increased or diminished an infinitely little. Hence it is, that they distinguish the Fluxion, or Difference of any Term, by the Letter *d*, as observed above, prefixed before it. Hence also this Method is called by them *Calculus differentialis*, the *Differential Method*. But this must not be confounded with another *Differential Method*, commonly called Sir *Isaac Newton's*; he being also the Inventor of that. It's a Method of drawing geometrical Curves through a Number of given Points, and is of great Use in Astronomical Purposes; but quite of another Kind from this.

XXV. Sir *Isaac Newton* delivered the Method of Fluxions according to both Conceptions of it, and frequently himself makes use of the latter; which is indeed, in many Instances, more convenient than the other; because what a Quantity gains by increasing, obtrudes itself upon the Mind sooner and more naturally than the Rate it increases at, does. What they call a Difference, is by him called an *Increment*, if the Quantity be increasing; and a *Decrement*, if it be decreasing: By which two Names, Increment and Decrement, is to be understood precisely the same Thing; they are accordingly by him called, by one common Name, *Moments*.

XXVI. In the Instances I have already given of the Application of this Doctrine, both the direct and inverse Methods are to be made use of; the Direct, in finding the Fluxion of a Quantity; the Inverse, in finding the Value of that Quantity from its Fluxion. But there are many other Uses of this Doctrine, some of which I shall now proceed to explain, in which the direct Method is alone sufficient; and consequently in which none of those Difficulties above-mentioned relating to the finding of Fluents, either by the Rules of this Method, or by the Tables above-mentioned, can occur, there being no Fluent in these Cases concerned, but what is already known and expressed. The Method of drawing Tangents, that is, of finding the Position of a Tangent to a curve Line, is of this Sort.

XXVII. The Principle on which the Method of doing this is founded, will be best explained by a Figure. While the Line PM, in Fig. 3 or 5. moves forward, describing the Area ARB in the manner above-mentioned, by its lengthening or shortening, as the Nature of the Figure requires, its Extremity M traces out the Curve AMR, and, by so doing, necessarily points out the Position or Situation of a Tangent to each Point of the Curve, as it passes through that Point. Thus, let ABC in Fig. 9. represent a Semicircle described by the Motion of PM: When PM is at

at QR, its Extremity R, by tracing out the Curve at R, marks out the Direction of a Tangent, as TR, to that Point; because the Tangent TR and the Curve at the Point R, where it touches it, lie both in the same Direction. When PM is passing over the Center at O, PM, now OB, neither lengthens, nor shortens, and its Extremity M is at that Instant moving along a Tangent, as HX, to that Point B, and by neither approaching towards the Diameter AC, nor departing from it, implies that that Tangent is parallel to the Diameter AC. When PM is come to DE, it decreases, and its Extremity, by tracing out the Curve at E, does at the same Time mark out the Position of the Line EY, a Tangent to that Point. Thus there is a manifest Relation between the Velocity the Perpendicular PM increases or decreases with, as it moves along, at every Point of the Curve, and the Position of a Tangent to the Curve at that Point. Which Relation is the Principle on which the Method of finding the Position of a Tangent to any Point of a Curve, in the fluxionary Way, is founded.

To shew how this is done, let it be considered, that, while the Line PM moves forwards, its Extremity M, which describes the Curve, is carried by two Motions at the same Time, in every Place except at the Point B, where that Line neither lengthens nor shortens. Suppose, for Instance, PM now come

40 *A preliminary Discourse.*

to QR ; then is the Point M , or R , carried by two Motions at the same Time, *viz.* one towards S , in the Line QR produced, by the lengthening of that Line; and another towards N , in a Direction parallel to QO , by the Motion of the Line QR towards OB . Represent therefore, and set off the Velocity you suppose QR to move with, by a Line as RN parallel to QO ; and compute the Velocity the Line PM shall lengthen with, that is, compute the Fluxion of the Line PM the Instant it comes to QR , which is readily done by the Rules of the *Direct* Method of Fluxions; set this off by a Line, as RS , and compleat the Parallelogram SN , and draw the Diagonal ZR : Then, because the Point R is carried forwards by a Motion, as RN ; and is at the same Time carried towards S with the Velocity RS , it is obvious, that, according to the well known Rules of Composition of Motion, the Point R is at this Instant moving in the Direction RZ , the Diagonal of that Parallelogram SN ; ZR is therefore a Tangent to the Curve at the Point R . Produce ZR till it meets the Diameter CA produced also in some Point, as T : Then will the Triangles ZNR and RQT be similar, the Sides of the one being respectively parallel to those of the other; say therefore, as SR or ZN , the computed Rate the Perpendicular QR increases at, is to NR , the assumed or supposed Rate it moves forwards with, so is QR to a fourth Term; which

A preliminary Discourse. 41

which will be QT , the Distance from Q , where a Tangent to the Point R must cut the Diameter CA , produced, if need be, beyond the Point A . Thus we have a general and ready Method of drawing Tangents to curve Lines: Which is of great Use in Mathematics.

XXVIII. The following is another Instance where the direct Method is alone sufficient. It relates to what is called in Geometry the determining the *Maxima* and *Minima*; by which is meant the discovering how large a Quantity must be taken, that another Quantity depending upon it shall be of the largest or least possible Magnitude; than which there is scarcely any Part of Geometry more useful, or entertaining. In order to understand it, it is to be observed, that there are some Quantities of such sort, that when the flowing or variable Term therein shall continually increase, the Quantities themselves shall not do so too, but shall first increase, and then decrease, as was hinted before: And there are others of such Kind, that while the flowing Term continues to increase, they shall first decrease, and then increase. Now this Doctrine teaches in what State, that is, how great or how small the variable Term in such Quantities is to be taken, that the Quantities themselves may be the largest possible in one Case, or the least possible in the other. For, as the first Sort first increase, and then decrease, it is manifest

42 *A preliminary Discourse.*

manifest there will be a Time when they shall be the greatest of all; and as the other first decrease, and afterwards increase, there will be a Time when they shall be the least of all. Thus, a Quantity, as $ax - xx$, is of this Kind; for if you suppose the Term a in this Quantity constant, and x variable, and take x larger and larger continually, you will find that Quantity $ax - xx$ shall grow larger and larger for a Time, and after that, it shall grow less and less, till it vanishes or becomes equal to Nothing. Now by this Branch of the fluxionary Method, it is discoverable how large the variable Term in such Quantities is to be taken, that the Quantities themselves may be as large as possible; and in Cases where they first decrease and then increase, how that Term is to be taken, that they may be the least of all. This is done by putting the Quantity into Fluxions, that is, finding, by the Rules of the direct Method, the Fluxion of the Quantity, and making it equal to Nothing; on this Principle, *viz.* that the Instant a Quantity is at the largest, or at the least, it neither increases nor decreases, but is, as it were, upon the Turn, and flows not at all. In the Instance before us, the Fluxion of the Quantity $ax - xx$, (supposing x to flow at the Rate $\dot{x}$) will be found by the Rules above-mentioned, to be $a\dot{x} - 2x\dot{x}$; making therefore this Expression equal to Nothing; that is, making it one Side of an Equation, and Nothing the other (thus $a\dot{x} - 2x\dot{x} = 0$)
implies

implies, that though x continues to increase on, at the same Rate $\dot{x}$, yet $a\dot{x} - 2x\ddot{x}$ the Fluxion of the Quantity $ax - xx$ is Nothing, or that the Quantity itself is at a Stand; and consequently, that it is as large as possible, and beginning to decrease; or that it is as small as possible, and beginning to increase. Then, by reducing that Equation $a\dot{x} - 2x\ddot{x} = 0$, or $a\dot{x} = 2x\ddot{x}$, in the usual Manner, we shall get $x = \frac{a}{2}$; which shews, that the Quantity $ax - xx$ is either the largest, or the least that is possible, when x is equal to $\frac{a}{2}$ or half a ; and that it is then the largest of all, may easily be tried, by putting it into Numbers, and taking x of several different Values successively.

XXIX. This Method we can apply to Mechanics, and discover thereby in what Circumstances a given Cause will produce the greatest Effect; or how the Parts of a Machine ought to be adjusted and proportioned, that it may perform the most Work in a given Time; or, which is a Consequence thereof, a given Quantity of Work with the least Expence: A Part of Science, not only extremely curious in itself, and a great Addition, but even an Ornament to Mechanics; and without which no Treatise on that Subject can be compleat. For an Instance of this, the Engine for raising Water by Fire being the grandest,

44 *A preliminary Discourse.*

grandest, and of the most Importance of any we have, and the Process being extremely easy, I will just shew how the principal Parts of that Machine ought to be proportioned, that its Effect in raising Water may be the greatest.

Let ABC , Fig. 10. represent the great Beam, or Lever, in that Engine turning on its Center B : Let p represent the Weight of the Atmosphere, or Air, pressing upon the Piston in the Cylinder, and x the Weight of a Column of Water to be raised at one Stroke: Then the Excess of p above x , that is, $p - x$, will express the Power p has to move the Machine, or to raise that Column of Water; that divided by $p + x$ (the Air and Water to be moved) gives $\frac{p - x}{p + x}$, which will be as the

Velocity of p , and consequently, (if AB be equal to BC) as the Velocity with which the Water will be raised; this multiplied by the Quantity of Water raised, that is, by x , will be the Momentum of the Water, or the Operation of the Machine, and will be $\frac{px - xx}{p + x}$,

which, that the Effect of the Machine may be the greatest, must be a *Maximum*. Imagine it to flow: Its Fluxion will be found, by the Rules of the direct Method, to be $pp\dot{x} - 2p\dot{x}x + p\dot{x}x - 2x\dot{x}x - p\dot{x}x + x\dot{x}x$

$\frac{p\dot{x} - x\dot{x}}{p + x}$ making

making which equal to Nothing, we have

$$\frac{pp\dot{x} - 2p\dot{x}x + p\dot{x}x - 2xxx - p\dot{x}x + xxx}{p + x^2} = 0.$$

Which, multiplying all by $p + x^2$ and dividing by $\dot{x}$, presently reduces itself to $pp - 2px - xx = 0$, or $xx + 2px = pp$, a common quadratic Equation, whose two Roots will be found to be $\pm \sqrt{2} \times p - p$, the larger of which is $+\sqrt{2} \times p - p$, that is, $0,4p$ nearly. If therefore, exclusive of Friction, and some other Considerations of small Importance, Matters be so ordered, that the Weight of Water in the Pumps be four tenths of that of the Air upon the Piston in the Cylinder, that Machine will perform the greatest Work in a given Time, or the same Work with the least Fewel, Machinery, and Attendance.

XXX. The Fluxions we have hitherto been considering, *viz.* the Velocities Quantities *increase* or *decrease* with, are called *first Fluxions*; besides which there are others, which are called, *second, third, fourth, fifth, &c.* Fluxions, even to Infinity; of which I must now give some Account. A *second Fluxion* then is not the Rate at which a Quantity increases, but the Change or Alteration that Rate of Increase may undergo: It does not, as the first, respect the *Quantity*, but the *Manner* of Increase or Decrease: It is not the Rate of Increase subsequent to the Change, but the
Change

46 *A preliminary Discourse.*

Change itself: The Rate of Increase subsequent to the Change, is a first Fluxion, as well as that which went before it, only larger or less than the first, as the Case may happen: The second Fluxion subsists wholly in the Change itself, and is greater or less according as the Change is greater or less. Accordingly when a Quantity increases with an uniform or even Pace, there is no second Fluxion of that Quantity. Its Rate of Increase is its first Fluxion; and as that Increase is continued on without Change or Interruption, there is no second Fluxion of that Quantity. But if the Quantity increases with an *uneven* Pace; if it changes its Rate of increasing from slower to quicker, or from quicker to slower, that single Circumstance of changing its Rate of increasing, is the *second Fluxion* of that Quantity. The following Instance will make this plainer. Let ABC, in Fig. 9. be a Semicircle, whose Center is O; and let the Base AP flow or increase uniformly; then, as is evident, the Perpendicular PM, moving along with the Point P, will increase slower and slower all the Way, till it comes to OB, where it will not increase at all. This implies a continual Alteration of its Rate of Increase. There is no Change or Alteration in the Rate of Increase of the Base in this Supposition; because we suppose it to flow with an *even* Pace; the Base therefore has no second Fluxion, but the Perpendicular PM has both first and second
all

all the Way; for it is always increasing, and always changing its Rate of increasing. The like is to be said of it in its Passage through the remaining Half of the Figure, except that it decreases there, as it increased here.

Let A M B, in Fig. 9. represent a Portion of any Curve, and suppose that the describing Line P M, when come to Q R, should from that Time continue to lengthen, as it moves along, at the same Rate it does there; then, as is most obvious, its Extremity R would continue in the Tangent R Z, and not R D, a Portion of the Curve, but R Z, a Portion of the Tangent T Z would be described by it: That its Extremity therefore may depart from the Tangent at R, in order to continue in the Curve R D, and to move along that, the Perpendicular Q R must necessarily abate of the Rate of Increase it had at R; which Alteration or *Abatement* is its second Fluxion there. The first Fluxion of the Perpendicular shews or determines the Situation of the Tangent T Z, as was explained before; the second Fluxion of the same, shews how far the Arch deviates from the Tangent at the Point of Contact. So that the Curvature or Bending of the Figure at every Point thereof, depends always on the *second Fluxion* of the Perpendicular at that Point.

XXXI. On this Principle is founded a most curious and ready Method of finding the exact Curvature a Curve has at any Point of it: For,
it

48 *A preliminary Discourse.*

it is only finding a Circle of such Radius, that its Ordinate, that is, the Perpendicular P M, so often mentioned, shall have its second Fluxion equal to the second Fluxion of the Ordinate of the Curve at that Point, when the first Fluxions of the Base and Ordinate are the same in each Figure; for then the Circle will have the same Degree of Deviation or Departure from its Tangent, that the Curve has at that Point; that is, it will be equally crooked with it there. A Problem of singular Use in the higher Geometry. For then that Curve may be considered, as to that Point of it, as if it were a Circle of such Radius; and may in all Respects be treated accordingly. Instances of which we shall have afterwards.

XXXII. As a second Fluxion is the Change or Alteration in the Velocity a Quantity increases or decreases with, that is, in the first Fluxion of that Quantity; so a *Third Fluxion* is the Change or Alteration in the second; and a *Fourth Fluxion* is the Change or Alteration in the third; and so on, *ad infinitum*. Second, third and fourth, &c. Fluxions therefore, and all the subsequent Orders of them, may be said to be the Changes and Subchanges which the Rates Quantities increase or decrease at, may undergo. The Characters they are generally represented by in the *New-*

tonian Way are, $\ddot{x}$, $\dot{\ddot{x}}$, $\ddot{\ddot{x}}$, &c. in the *Leib-*
nitzian, ddx , $dddx$, d^4x , &c. I shall say
nothing

nothing more of them in this Place; but, presuming the Reader has now a tolerable Notion of what is meant by the Method of Fluxions, which is as much as was intended by this preliminary Discourse, shall proceed to consider the Usefulness, or rather the Necessity of that Method in the Business of Natural Philosophy, to which I shall subjoin a short Account of the Controversy referred to in the Title: Which will farther conduce towards rendering his Conception of the Nature of this Doctrine still more clear and express.

XXXIII. I observed at the Beginning of this Discourse, that, in Consequence of that Facility and Readiness wherewith curvilinear Figures are treated by this Method, a large Field for Discoveries and Improvements in Natural Philosophy was laid open: I shall now explain myself on that Head, by laying before the Reader an Example or two to shew, how the Consideration of such Figures, particularly their Areas, conduces towards some of the most important Discoveries in that Science.

It is demonstrated by Sir *Isaac Newton*, (*Principia Mathematica*, Lib. 1. Prop. 39.) that if a Body, as A, (see Fig. 11.) be attracted towards a distant Point C, with the same or with different Degrees of Force at different Distances from C; and the Body be supposed to move freely towards C, solely by these At-

E

tractions,

tractions, and the Degree of Force it is attracted with when at A, be represented and set off at A by the Perpendicular AB; and the Force the Body shall be attracted with when it comes to D, be represented and set off by the Perpendicular DH; (that is, if DH be drawn so many times longer or shorter than AB, as the Force of Attraction at D is greater or less than the Force that acted at A;) and in like manner, if the several Degrees of Force that act at the several Points E, F and G, &c. be represented and set off by the Perpendiculars EI, FK, GL, &c. and the Line BHIKL be drawn through the Extremities of those Perpendiculars, as represented in the Figure; then the Velocity the Body A shall have when it comes to D, shall be to its Velocity at any other Point, as F, as the square Root of the Area ADHB to the square Root of the Area AFKB; and so for any other Places in the Line AC. He demonstrates also, in the same Proposition, that if the Perpendiculars DM, EN, FO, &c. were taken reciprocally proportionable to the Velocity the Body shall be found to have at the several Points D, E, F, &c. and be set off at those Points, as in the Figure, and the Line AMNO were drawn through the Extremities of them, the Area ADM would be to the Area AEN, as the Time the Body would take up in coming to D, to the Time it would take up in coming to E; and the like for

for any other Places in the Line A C. If therefore the attracting Powers at all Distances from the Point C, and the Velocity with which the Body A, when moved by these Forces only, shall come to any one Point, as D, be known; then the Velocity it shall have, when it comes to any other Point between A and C, may be determined by this Proposition: And if the Time the Body shall take up in coming to any one Place as D, be known, the Time in which it shall come to any other Place between A and C, may be found from hence. A fundamental Proposition this in Natural Philosophy, and which paves the Way to some of the greatest Discoveries in that Science. But unless those curvilinear Areas A G L B, and A G P, or any Part of them, could be measured, this Proposition would be insignificant. This is the Reason Sir *Isaac Newton*, in laying down this Proposition and many other in his *Principia*, uses these Words, *Concessis figurarum curvilinearum quadraturis*; that is, allowing that curvilinear Figures can be measured. Hence we see the Use of the Mensuration, or *Quadrature*, as they term it, of Areas bounded by curve Lines; and consequently the great Expediency of the Fluxionary Method; because by it those Areas are so readily measured.

Again, when that great Author is laying down Rules for estimating the Powers wherewith Bodies attract one another, he shews in

52 *A preliminary Discourse.*

the 90th Proposition of the first Book of his *Principia*, that if a Particle of Matter, as P, in Fig. 12. be attracted towards a circular Plane, as DCB, by each Particle in that Plane, all the Particles thereof having the same or equal Degrees of Attraction; and the Particle P be situated in a Line AP, perpendicular to the Plane, and passing through its Center at A; and if PH be taken equal to PD, and the Force wherewith the Particle P is attracted towards A by the Point D, be computed and set off at H by the Perpendicular HI; and PF be made equal to PE, and the Force the Particle P is attracted with towards A by the Point E be computed also, and set off at F, by the Perpendicular FK; and in like manner, if a Perpendicular be set off upon the Line AH, representing the Force with which the Particle P is attracted towards A by each Point in the whole Semidiameter DA; and the Line IKL be drawn through the Extremities of all the Perpendiculars, and the whole Area AHIL be multiplied by the Distance AP, the Product that shall arise from that Multiplication, will express the *united* Force wherewith all the Particles in the whole circular Plane DCB, shall attract the Particle P towards A the Center of the Plane.

These are Instances how such curvilinear Areas become subservient to the Purposes of Natural Philosophy; and point out the Uses
of

of the fluxionary Method by which their Magnitudes are to be determined.

XXXIV. In this Manner Natural Philosophers represent the Quantities they would consider, as the Powers that shall act upon Bodies in such and such Circumstances, the Spaces they shall move over in Consequence thereof in certain Times; the Velocities they shall at any Time move with; the Times they shall take up in moving, and in general all Sorts of Causes and Effects, by geometrical Quantities, as Lines, Angles, Areas, &c. and then the finding the Values of those geometrical Quantities, shews how great those Spaces, how quick those Velocities, how powerful those Causes, &c. are. Hence it is that Geometry is so exceedingly useful in the Business of Natural Philosophy, and in particular the Method of Fluxions, which, if I may so say, renders that Geometry free and unconfined, extending it to Figures of all Sorts.

I shall, in the Course of this Work, I mean in the Treatise for which this preliminary Discourse is intended, shew particularly how the Magnitude of the above-mentioned Areas, and many other in the *Principia*, are to be found by the Method of Fluxions; and do propose to consider and explain several of the more remarkable Propositions in that Treatise, as also some of those of other Authors; for which Purpose I shall make them Examples to my Rules, as I go along, and

54 *A preliminary Discourse.*

take that Occasion to enlarge upon, and to illustrate them.

XXXV. The Doctrine of Fluxions, as hinted before, was ushered into the World by two very great Persons, Sir *Isaac Newton*, and *Leibnitz* a learned Mathematician abroad. But the latter is generally thought to have received some Hints of it from the former. They each explained it a different Way. Each of them had their Followers; who, for the most part, adhered to the Method delivered by their respective Master. The *English* Mathematicians considered it, as delivered by Sir *Isaac*; most of the Foreigners, as set forth by *Leibnitz*. However, this Doctrine, as delivered and explained by them, was thought by some to be founded on Principles that were obscure; and even to fall short of Geometrical Rigour, that great Ornament of the mathematical Sciences. The first Person that objected to it publicly, was the late learned Dr. *Berkley* Bishop of *Cloyne*, in a Pamphlet he calls the *Analyst*; wherein he endeavours to shew the Doctrine of Fluxions, as treated by the above-mentioned Authors and their Followers, to be both *obscure* and *unscientific*. This Pamphlet, coming from a Person of great Abilities, roused the Mathematicians, some of whom attempted to defend it, as delivered by Sir *Isaac*. Others declined entering the Combat; but endeavoured to treat it
in

in a Manner less exceptionable ; particularly the late Mr. *Robins* ; as also that profound Mathematician the late *Mac Laurin*, Professor of Geometry at *Edinburgh*, which latter has established the Rules of it, agreeably to the Rigour of the ancient Geometry, by Demonstrations of the strictest Form. But no Body, that I know of, has explained it in so easy and familiar a Way as I apprehend the Subject capable of. For this Reason, and for the sake, as I hinted just now, of explaining some of the more illustrious Propositions in the *Principia*, and in other Books, in my own Way, it is, that I undertake that Part of the Task.

The Manner in which the two great Authors above-mentioned laid down the Principles of their Methods, is as follows.

56 *A preliminary Discourse.*

SIR Isaac Newton has comprised the Grounds of his Method in one Lemma, viz. the second of the second Book of his *Principia*. In explaining which, he has these Words: “Has Quantitates [Quantitates scilicet Arithmeticas et Geometricas] ut indeterminate terminatas et instabiles, et quasi motu fluxuve perpetuo crescentes vel decrescetes, hic considero; et earum incrementa vel decrementa momentanea sub nomine momentorum intelligo: Ita ut incrementa pro momentis additiis seu affirmativis, ac decrementa pro subductiis seu negativis habeantur. Cave tamen intellexeris particulas finitas. Particulæ finitæ non sunt momenta, sed quantitates ipsæ ex momentis genitæ: Intelligenda sunt Principia jamjam nascentia finitarum magnitudinum. Neque enim spectatur in hoc Lemmate magnitudo momentorum, sed prima nascentium proportio. Eodem recidit, si loco momentorum usurpentur vel velocitates incrementorum ac decrementorum (quas etiam motus, mutationes et fluxiones quantitatum nominare licet) vel finitæ quævis quantitates velocitatibus hisce proportionales.” As much as to say, “This Sort of Quantities, [viz. Arithmetical and Geometrical ones] I consider as variable and indeterminate, always increasing or decreasing by a perpetual Flux: And their instantaneous Increments or De-

crements

“crements I call Moments; looking upon
“those Increments or Augmentations as affirmative Moments, or Moments to be added; and those Decrements or Diminutions, as negative Moments, or Moments to be deducted: But those Moments are by no means to be considered as finite Quantities; such are not Moments, but are Quantities arising from Moments. By Moments I would have understood the Elements of finite Magnitudes just rising out of Nothing, and before they become finite. I don't regard the Magnitude of those Moments, but only the Proportion they bear to each other the Instant they begin to exist. Or, instead of the Moments, we may use the Velocities wherewith the Increments or Decrements are generated, (which Velocities we may also call Motions, Changes and Fluxions of Quantities) or we may use any Quantities for them, that are proportionable to those Velocities.”

XXXVI. He farther explains himself on this Head, in the Introduction to his Book of Quadratures, in these Words: “Fluxiones sunt quam proximè ut fluentium augmenta æqualibus temporis particulis quam minimis nascentium; et, ut accuratè loquar, sunt in prima ratione Augmentorum nascentium; exponi autem possunt per líneas quascunque quæ sunt ipsis proportionales.” That is, “The Fluxions of Quantities are
“very

58 *A preliminary Discourse.*

“ very nearly as the Increments of their Flu-
 “ ents generated in the least equal Particles of
 “ Time; and they are exactly in the first
 “ Proportion of the said Increments; *viz.* in
 “ the Proportion they bear to each other, the
 “ Instant they arise out of Nothing: And
 “ they may be expressed, or denoted, by any
 “ Lines that are proportionable to those In-
 “ crements.”

XXXVII. In the *Leibnitzian* Way, Quan-
 tities are not considered as increasing or de-
 creasing gradually, but as it were *per Saltum*:
 They are supposed to acquire their Augmen-
 tations or Diminutions piece by piece; which
 Augmentations or Diminutions are, as I ob-
 served before, called Differences: And those
 Differences are supposed to be infinitely smaller
 than the Quantities whose Differences they
 are. Those Differences have also their Aug-
 mentations and Diminutions, which are call-
 ed the Differences of the first Differences.
 These are supposed to be infinitely smaller
 than the former; and so on, without End.
 The Differences of the first Differences answer
 to second Fluxions in the *Newtonian* Way;
 the Differences of those, to third Fluxions;
 and so on.

From this Way of conceiving and expres-
 sing this Doctrine, it is called by the foreign
 Mathematicians, *Calculus differentialis*; the
Differential Method; the *Method of Infinitesi-*
mals;

mals; and by the French, *Analyse des Infiniment Petits*.

XXXVIII. The Differences of *Leibnitz* are what Sir *Isaac Newton* in the *Lemma* calls instantaneous Increments or Decrements; and by one general Name, Moments; which Moments are there represented, as Something that may be added, Something that may be deducted, and yet not finite Quantities; as infinitely little, and as not actually existing, but beginning to exist; as Elements of finite Quantities just rising out of Nothing, and before they become finite; and as being capable of Proportion: To which it is added, that the Velocities, or any Quantities proportionable to the Velocities they are generated with, may be used in their stead. Which Conceptions are severely censured in the *Analyst*.

XXXIX. The ingenious Author of that Pamphlet, after having observed, " That
" Geometry is an excellent Logic: That it
" must be owned, that when the Definitions
" are clear; when the Postulata cannot be
" refused, nor the Axioms denied; when
" from the distinct Contemplation and Com-
" parison of Figures, their Properties are de-
" rived by a perpetual well-connected Chain
" of Consequences, the Objects being still
" kept in View, and the Attention ever fixed
" upon them; there is acquired an Habit of
" reasoning, close and exact and methodical:
" Which Habit strengthens and sharpens the
" Mind,

60 *A preliminary Discourse.*

“ Mind, and being transferred to other Sub-
“ jects, is of general Use in the Inquiry after
“ Truth.” He then goes on to examine,
how far this may be the Case, as he expresses
it, of our Geometrical Analysts, in the fol-
lowing Words.

“ The Method of Fluxions is the general
“ Key, by Help whereof the modern Mathe-
“ maticians unlock the Secrets of Geometry,
“ and consequently of Nature. And, as it is
“ that which hath enabled them so remarkably
“ to outgo the Ancients in discovering Theo-
“ rems and solving Problems, the Exercise
“ and Application thereof is become the main,
“ if not sole, Employment of all those who
“ in this Age pass for profound Geometers.
“ But whether this Method be clear or ob-
“ scure, consistent or repugnant, demonstra-
“ tive or precarious, as I shall inquire with
“ the utmost Impartiality, so I submit my
“ Inquiry to your own Judgment, and that of
“ every candid Reader. Lines are supposed
“ to be generated by the Motion of Points,
“ Plains by the Motion of Lines, and Solids
“ by the Motion of Plains. And whereas
“ Quantities generated in equal times are
“ greater or lesser, according to the greater or
“ lesser Velocity, wherewith they increase and
“ are generated, a Method hath been found
“ to determine Quantities from the Velocities
“ of their generating Motions. And such
“ Velocities are called Fluxions: And the
“ Quantities

“ Quantities generated are called flowing
“ Quantities. These Fluxions are said to be
“ nearly as the Increments of the flowing
“ Quantities, generated in the least equal Par-
“ ticles of Time; and to be accurately in the
“ first Proportion of the nascent, or in the
“ last of the evanescent, Increments. Some-
“ times, instead of Velocities, the momen-
“ taneous Increments or Decrements of unde-
“ termined flowing Quantities are considered,
“ under the Appellation of Moments.

“ By Moments we are not to understand
“ finite Particles. These are said not to be
“ Moments, but Quantities generated from
“ Moments, which last are only the nascent
“ Principles of finite Quantities. It is said,
“ that the minutest Errors are not to be neg-
“ lected in Mathematics: That the Fluxions
“ are Celerities, not proportional to the finite
“ Increments though ever so small; but only
“ to the Moments or nascent Increments,
“ whereof the Proportion alone, and not the
“ Magnitude, is considered. And of the
“ aforesaid Fluxions there be other Fluxions,
“ which Fluxions of Fluxions are called se-
“ cond Fluxions. And the Fluxions of these
“ second Fluxions are called third Fluxions;
“ and so on, fourth, fifth, sixth, &c. *ad in-*
“ *finitum*. Now as our Sense is strained and
“ puzzled with the Perception of Objects ex-
“ tremely minute, even so the Imagination is
“ very much strained and puzzled to frame
“ clear

62 *A preliminary Discourse.*

“ clear Ideas of the least Particles of Time, or
 “ the least Increments generated therein :
 “ And much more so to comprehend the
 “ Moments, or those Increments of the flow-
 “ ing Quantities *in statu nascenti*, in their very
 “ first Origin or beginning to exist, before
 “ they become finite Particles. And it seems
 “ still more difficult to conceive the abstracted
 “ Velocities of such nascent imperfect Entities.
 “ But the Velocities of the Velocities, the se-
 “ cond, third, fourth, fifth Velocities, &c.
 “ exceed, if I mistake not, all human Un-
 “ standing. The farther the Mind analyseth,
 “ and pursueth these fugitive Ideas, the more
 “ it is lost and bewildered; the Objects at
 “ first fleeting and minute, soon vanishing
 “ out of Sight. Certainly in any Sense, a
 “ second or third Fluxion seems an obscure
 “ Mystery. The incipient Celerity of an in-
 “ cipient Celerity, the nascent Augment of a
 “ nascent Augment, *i. e.* of a Thing which
 “ hath no Magnitude. Take it in which
 “ Light you please, the clear Conception of
 “ it will, if I mistake not, be found impos-
 “ sible: Whether it be so or not, I appeal to
 “ the Trial of every thinking Reader. And
 “ if a second Fluxion be inconceivable, what
 “ are we to think of third, fourth, fifth
 “ Fluxions, and so onward without End?
 “ The foreign Mathematicians are suppo-
 “ sed by some, even of our own, to proceed
 “ in a Manner less accurate perhaps and geo-
 “ metrical,

“ metrical, yet more intelligible. Instead of
“ flowing Quantities and their Fluxions, they
“ consider the variable finite Quantities, as
“ increasing or diminishing by the continual
“ Addition or Subduction of infinitely small
“ Quantities. Instead of the Velocities where-
“ with Increments are generated, they con-
“ sider the Increments or Decrements them-
“ selves, which they call Differences, and
“ which are supposed to be infinitely small.
“ The Difference of a Line is an infinitely
“ little Line; of a Plain an infinitely little
“ Plain. They suppose finite Quantities to
“ consist of Parts infinitely little, and Curves
“ to be Polygons, whereof the Sides are infi-
“ nitely little, which by the Angles they
“ make one with another determine the Cur-
“ vity of the Line. Now to conceive a Quan-
“ tity infinitely small, that is, infinitely less
“ than any sensible or imaginable Quantity,
“ or than any the least finite Magnitude, is,
“ I confess, above my Capacity. But to con-
“ ceive a Part of such infinitely small Quan-
“ tity, that shall be still infinitely less than it,
“ and consequently though multiplied infi-
“ nitely shall never equal the minutest finite
“ Quantity, is, I suspect, an infinite Diffi-
“ culty to any Man whatever; and will be
“ allowed such by those who candidly say
“ what they think; provided they really
“ think and reflect, and do not take Things
“ upon Trust.

64 *A preliminary Discourse.*

“ And yet in the *Calculus differentialis*,
 “ which Method serves to all the same In-
 “ tents and Ends with that of Fluxions, our
 “ modern Analysts are not content to confi-
 “ der only the Differences of finite Quanti-
 “ ties; they also consider the Differences of
 “ those Differences, and the Differences of
 “ the Differences of the first Differences; and
 “ so on *ad infinitum*. That is, they consider
 “ Quantities infinitely less than the least dis-
 “ cernible Quantity; and others infinitely less
 “ than those infinitely small ones; and still
 “ others infinitely less than the preceding In-
 “ finitefimals, and so on without End or Li-
 “ mit. Infomuch that we are to admit an
 “ infinite Succession of Infinitefimals, each
 “ infinitely less than the foregoing, and infi-
 “ nitely greater than the following. As there
 “ are first, second, third, fourth, fifth, &c.
 “ Fluxions, so there are Differences, first,
 “ second, third, fourth, &c. in an infinite
 “ Progression towards Nothing, which you
 “ still approach and never arrive at. And
 “ (which is most strange) altho’ you should
 “ take a Million of Millions of these Infi-
 “ nitefimals, each whereof is supposed infinite-
 “ ly greater than some other real Magnitude,
 “ and add them to the least given Quantity,
 “ it shall be never the bigger. For this is one
 “ of the modest *Postulata* of our modern Ma-
 “ thematicians, and is a Corner-stone or
 “ Ground-work of their Speculations.

“ It

“ It must indeed be acknowledged, the
 “ modern Mathematicians do not consider
 “ these Points as Myſteries, but as clearly con-
 “ ceived and maſtered by their comprehen-
 “ ſive Minds. They ſcruple not to ſay, that
 “ by the Help of theſe new Analytics they
 “ can penetrate into Infinity itſelf: That they
 “ can even extend their Views beyond Infi-
 “ nity: That their Art comprehends not only
 “ Infinite, but Infinite of Infinite (as they
 “ expreſs it) or an Infinity of Infinities. But,
 “ notwithſtanding all theſe Assertions and Pre-
 “ tenſions, it may be juſtly queſtioned whe-
 “ ther, as other Men in other Inquiries are
 “ often deceived by Words or Terms, ſo they
 “ likewise are not wonderfully deceived and
 “ deluded by their own peculiar Signs, Sym-
 “ bols, or Species. Nothing is eaſier than to
 “ deviſe Expreſſions or Notations for Fluxions
 “ and Infinitesimals of the firſt, ſecond, third,
 “ fourth and ſubſequent Orders, proceeding
 “ in the ſame regular Form without End or
 “ Limit, $x, x, x, x, &c.$ or $dx, ddx, dddx,$
 “ $ddddx, &c.$ Theſe Expreſſions indeed are
 “ clear and diſtinct, and the Mind finds no
 “ Difficulty in conceiving them to be continu-
 “ ed beyond any assignable Bounds. But if we
 “ remove the Veil and look underneath, if
 “ laying aſide the Expreſſions we ſet ourſelves
 “ attentively to conſider the Things them-
 “ ſelves, which are ſuppoſed to be expreſſed

66 *A preliminary Discourse.*

“ or marked thereby, we shall discover much
 “ Emptiness, Darkness and Confusion; nay,
 “ if I mistake not, direct Impossibilities and
 “ Contradictions. Whether this be the Case
 “ or no, every thinking Reader is intreated
 “ to examine and judge for himself.”

XL. After this he proceeds to inquire into the Proofs by which those great Authors and their Followers established the Rules of the fluxionary Method. But before I lay down his Objections to them, I shall say something preparatory to the understanding those Proofs.

XLI. The fundamental Point in discovering the Rules to be made use of in the fluxionary Method, is, when a Quantity, or Number, is supposed to increase, to find how fast the Square, the Cube, or any other Power of that Quantity, shall increase in Consequence thereof.

But before I lay down the Method they make use of for this Purpose, it is proper to observe, that when any Number or Quantity increases uniformly, every Power of that Quantity increases with a Motion that is accelerated or retarded: The Square, for Instance, shall increase faster and faster; the square Root, slower and slower. To instance in the Square of the Number 3; while the Number 3 increases to 4, the Square of it, viz. 9, increases to 16; but while the Number itself increases from 4 to 5, the Square increases from

16 to 25, which is a greater Increase than the former; while it increases from 5 to 6, the Square increases from 25 to 36, which is a greater Increase than the last. From whence it is evident, that while a Quantity flows with an uniform or even Pace, the Square of that Quantity increases faster and faster perpetually.

The like is to be said of other Powers: They increase with variable Motions, when the Quantities, whose Powers they are, increase with uniform ones. Hence it is, that their Increments, or what they gain by increasing, are not proper Measures of the Rates they begin to increase with, unless those Increments be taken infinitely small; that is, unless the Time they arise in be taken infinitely short. This is the Reason those great Authors considered the Augmentations, Increments or Decrements, Moments or Differences of Quantities, as infinitely small.

XLII. These Things being observed, the Method they made use of for demonstrating the Rules, will be more readily understood.

The usual Way is to demonstrate for all Powers at once by one general Process; but as such general Process may not be so readily understood, I shall prepare the Reader for it, by shewing it first in the finding the Fluxion of a single Power only; for Instance, the Cube, or third Power.

68 *A preliminary Discourse.*

XLIII. Let a Quantity then, which we will call x , flow or increase; and let it be proposed to find at what Rate the Cube of that Quantity shall flow at the same Time.

Let us suppose that the Quantity x by flowing becomes $x + o$; but observe, that the Letter o here is not put to denote Nothing, but such Quantity as x may be supposed to acquire by increasing during any indeterminate Time: Then the Cube of $x + o$ being $xxx + 3xxo + 3xoo + ooo$, the Cube of x , which before the Increase began was only xxx , is now $xxx + 3xxo + 3xoo + ooo$; that is, it is larger than before by $3xxo + 3xoo + ooo$. This therefore is the temporary Increment of xxx , or the Augmentation it acquires, while the Quantity itself is increased by the finite Increment o . The Proportion therefore which those Increments bear to each other, is that of o to $3xxo + 3xoo + ooo$. But, agreeably to what was observed about the Increment of a Square, or other Power, the Proportion those Increments bear to each other, is not the true Proportion of the Rates the Quantities x and xxx increase with at first setting off, or at the Beginning of the Increase, unless they be taken such as would arise in an infinitely short Time; that is, unless the Quantity o be considered as just coming into Being, or as infinitely little. Let it be so; then the last Term of the Increment of the Cube, *viz.* ooo , which consists of the infinitely

nitely little Quantity o multiplied by itself twice, is infinitely less than $3xoo$, in which o is multiplied by itself but once; for this Reason they reject that as Nothing. Again, the Term $3xoo$, where o is multiplied by itself once; is infinitely less than $3xxo$, where it is not multiplied by itself at all; they therefore reject that also as Nothing: And then there is left only o and $3xxo$ to express the nascent or evanescent Increments of the Quantities x and xxx , or the Rates those Quantities respectively begin to increase with. Now the Proportion those Terms o and $3xxo$ bear to each other, is evidently that of Unity to $3xx$; because when each of them is divided by the Term o , they give 1 and $3xx$. If therefore we express the Rate any Quantity increases with by Unity, the Term $3xx$, that is, thrice the Square of that Quantity, shall denote the Rate the Cube thereof shall increase with at the same Time,

There is another Way of conducting this Process, the same in Reallity, though different in Appearance; which is thus. When the finite Increments are found to be o and $3xxo + 3xoo + ooo$, as above, to divide each Term immediately by o , by which means they become 1 and $3xx + 3xo + oo$; and then to suppose the Increments to vanish, or the Quantity o to be infinitely small; upon which the two last Terms, viz. $3xo$ and oo , go out as Nothings, being infinitely smaller

70 *A preliminary Discourse.*

than $3xx$; and we have only left 1 and $3xx$, as before.

XLIV. These are the Methods by which the Fluxions of Powers are discovered, only with this Difference; *viz.* that instead of inquiring after the Fluxion of a particular Power, as x cubed, or raised to the third Power, they inquire after that of x raised to the Power n , by which means the Process becomes general, and serves for any Power whatever. See it represented by the Analyst, as follows.

“ Now the Method of obtaining a Rule to
 “ find the Fluxion of any Power, is as fol-
 “ lows. Let the Quantity x flow uniformly,
 “ and be it proposed to find the Fluxion of
 “ x^n . In the same time that x by flowing
 “ becomes $x + o$, the Power x^n becomes
 “ $\overline{x + o}^n$, that is, by the Method of infinite
 “ Series, (or what is called Sir *Isaac Newton's*
 “ Binomial Theorem) $x^n + nox^{n-1} + \frac{nn-n}{2}$
 “ $oox^{n-2} +$, &c. and the Increments o and
 “ $nox^{n-1} + \frac{nn-n}{2}oox^{n-2}$, &c. are one to
 “ another, as 1 to $nx^{n-1} + \frac{nn-n}{2}ox^{n-2} +$,
 “ &c. Let now the Increments vanish, and
 “ their last Proportion will be 1 to nx^{n-1} .
 “ But it should seem that this Reasoning is
 “ not

“ not fair and conclusive. For when it is
 “ said, let the Increments vanish, that is, let
 “ the Increments be Nothing, or let there be
 “ no Increments, the former Supposition that
 “ the Increments were something, or that
 “ there were Increments is destroyed, and yet
 “ a Consequence of that Supposition, that is,
 “ an Expression got by Virtue thereof, is re-
 “ tained; which is a false Way of reasoning.
 “ Certainly when we suppose the Increments
 “ to vanish, we must suppose their Propor-
 “ tions, their Expressions, and every Thing
 “ else derived from the Supposition of their
 “ Existence to vanish with them.

“ To make this Point plainer, I shall un-
 “ fold the Reasoning, and propose it in a
 “ fuller Light to your View. It amounts
 “ therefore to this, or may in other Words
 “ be thus expressed. I suppose that the
 “ Quantity x flows, and by flowing is in-
 “ creased, and its Increment I call o , so that
 “ by flowing it becomes $x + o$. And as x in-
 “ creaseth, it follows that every Power of x
 “ is likewise increased in due Proportion.
 “ Therefore as x becomes $x + o$, x^n will be-

“ come $x + o^n$; that is, according to the Me-
 “ thod of infinite Series $x^n + n o x^{n-1}$

“ $+ \frac{nn-n}{2} oo x^{n-2} +, \&c.$ And if from the

“ two augmented Quantities we subduct the
 “ Power and the Root respectively, we shall

72 *A preliminary Discourse.*

“ have remaining the two Increments, to

“ wit, o and $nx^{n-1} + \frac{nn-n}{2} oo x^{n-2} +, \&c.$

“ Which Increments being both divided by
“ the common Divisor o , yield the Quotients

“ 1 and $nx^{n-1} + \frac{nn-n}{2} o x^{n-2} +, \&c.$ which

“ therefore express the Ratio of the Incre-
“ ments. Hitherto I have supposed that x
“ flows, that x has a real Increment, that o
“ is something. And I have proceeded all
“ along on that Supposition, without which
“ I should not have been able to have made
“ so much as one single Step. From that
“ Supposition it is that I get at the Increment
“ of x , that I am able to compare it with the
“ Increment of x , and that I find the Propor-
“ tion between the two Increments. I now
“ beg Leave to make a new Supposition con-
“ trary to the first, that is, I will suppose
“ that there is no Increment of x , or that o is
“ Nothing; which second Supposition de-
“ stroys my first, and is inconsistent with it,
“ and therefore with every Thing that sup-
“ poseth it. I do nevertheless beg Leave to
“ retain nx^{n-1} , which is an Expression ob-
“ tained in Virtue of my first Supposition,
“ which necessarily presupposeth such Suppo-
“ sition, and which could not be obtained
“ without it: All which seems a most incon-
“ sistent Way of arguing.

“ Nothing

“ Nothing is plainer than that no just Con-
“ clusion can be directly drawn from two in-
“ consistent Suppositions. You may indeed
“ suppose any thing possible: But afterwards
“ you may not suppose any thing that de-
“ stroys what you first supposed. Or if you
“ do, you must begin *de novo*. If therefore
“ you suppose that the Augments vanish, that
“ is, that there are no Augments, you are to
“ begin again, and see what follows from
“ such Supposition. But Nothing will follow
“ to your Purpose. You cannot by that
“ means ever arrive at your Conclusion, or
“ succeed in what is called by the celebrated
“ Author, the Investigation of the first or
“ last Proportions of nascent and evanescent
“ Quantities, by instituting the Analysis in
“ finite ones. I repeat it again: You are at
“ Liberty to make any possible Supposition;
“ and you may destroy one Supposition by
“ another: But then you may not retain the
“ Consequences, or any Part of the Conse-
“ quences, of your first Supposition so destroy-
“ ed. I admit that Signs may be made to
“ denote either any thing, or nothing: And
“ consequently that in the original Notation
“ $x + o$, o might have signified either an In-
“ crement or Nothing. But then which of
“ these soever you make it signify, you must
“ argue consistently with such its Signification,
“ and not proceed upon a double Meaning:
“ Which to do were a manifest Sophism. Whe-
“ ther

74 *A preliminary Discourse.*

“ ther you argue in Symbols or in Words,
 “ the Rules of right Reason are still the
 “ same. Nor can it be supposed you will
 “ plead a Privilege in Mathematics to be
 “ exempt from them.

“ If you assume at first a Quantity increased
 “ by Nothing, and in the Expression $x + o$,
 “ o stands for Nothing, upon this Supposi-
 “ tion as there is no Increment of the Root,
 “ so there will be no Increment of the Power;
 “ and consequently there will be none except
 “ the first, of all those Members of the Series
 “ constituting the Power of the Binomial;
 “ you will therefore never come at your Ex-
 “ pression of a Fluxion legitimately by such
 “ Method. Hence you are driven into the
 “ fallacious Way of proceeding to a certain
 “ Point on the Supposition of an Increment,
 “ and then at once shifting your Supposition
 “ to that of no Increment. There may seem
 “ great Skill in doing this at a certain Point or
 “ Period. Since if this second Supposition
 “ had been made before the common Division
 “ by o , all had vanished at once, and you
 “ must have got nothing by your Suppo-
 “ sition. Whereas by this Artifice of first
 “ dividing, and then changing your Suppo-
 “ sition, you retain 1 and nx^{n-1} . But not-
 “ withstanding all this Address to cover it,
 “ the Fallacy is still the same. For whether
 “ it be done sooner or later, when once the
 “ second Supposition or Assumption is made,
 “ in

“ in the same Instant the former Assumption,
“ and all that you got by it, is destroyed,
“ and goes out together. And this is univer-
“ sally true, be the Subject what it will,
“ throughout all the Branches of human
“ Knowledge; in any other of which, I be-
“ lieve, Men would hardly admit such a Rea-
“ soning as this, which in Mathematics is ac-
“ cepted for Demonstration.”

XLV. The Analyst objects also to the Method made use of by Sir *Isaac Newton* in his Demonstration of his above-mentioned Lemma,* for obtaining the Fluxion or Moment of a Rectangle or Product of two indeterminate Quantities; from whence he deduces Rules for finding the Fluxions of all other Products and Power, whether the Coefficients or the Indexes be Integers or Fractions, rational or surd.

“ This fundamental Point, says he, one
“ would think should be very clearly made
“ out, considering how much is built
“ upon it; and that its Influence extends
“ throughout the whole *Analysis*. But let
“ the Reader judge. This is given for De-
“ monstration: Suppose the Product or Rect-
“ angle $A \times B$ increased by continual Motion,
“ and that the momentaneous Increments of
“ the Sides A and B are a and b . When the
“ Sides A and B were deficient, or lesser by
“ one

* *Principia Mathematica*, Lib. 2. Lem. 2.

76 *A preliminary Discourse.*

“ one Half of the Moments, the Rectangle
 “ was $A - \frac{1}{2}a \times B - \frac{1}{2}b$, that is, $AB - \frac{1}{2}aB$
 “ $- \frac{1}{2}bA + \frac{1}{4}ab$. And as soon as the
 “ Sides A and B are increased by the other
 “ two Halves of their Moments, the Rect-
 “ angle becomes $A + \frac{1}{2}a \times B + \frac{1}{2}b$, or AB
 “ $+ \frac{1}{2}aB + \frac{1}{2}bA + \frac{1}{4}ab$. From the latter
 “ Rectangle subduct the former, and the re-
 “ maining Difference will be $aB + bA$.
 “ Therefore the Increment of the Rectangle
 “ generated by the entire Increments a and b
 “ is $aB + bA$. *Q. E. D.* But it is plain
 “ that the direct and true Method to obtain
 “ the Moment or Increment of the Rectangle
 “ AB is to take the Sides as increased by
 “ their whole Increments, and so multiply
 “ them together, $A + a$ by $B + b$, the Pro-
 “ duct whereof $AB + aB + bA + ab$ is the
 “ augmented Rectangle; whence if we sub-
 “ duct AB, the Remainder $aB + bA + ab$
 “ will be the true Increment of the Rectangle,
 “ exceeding that which was obtained by the
 “ former illegitimate and indirect Method,
 “ by the Quantity ab . And this holds uni-
 “ versally be the Quantities a and b what they
 “ will, big or little, Finite or Infinitesimal,
 “ Increments, Moments or Velocities. Nor
 “ will it avail to say, that ab is a Quantity
 “ exceeding small: Since we are told, that
 “ *in rebus mathematicis errores quam minimi*
 “ *non sunt contemnendi.** Such Reasoning as
 “ this

* *Introductio ad Quadraturam Curvarum.*

“ this for Demonstration, nothing but the
“ Obscurity of the Subject could have en-
“ couraged or induced the great Author of the
“ Fluxionary Method to put upon his Fol-
“ lowers, and nothing but an implicit Defe-
“ rence to Authority could move them to ad-
“ mit. The Case indeed is difficult. There
“ can be nothing done till you have got rid
“ of the Quantity ab . In order to this, the
“ Notion of Fluxions is shifted: It is placed
“ in various Lights: Points which should be
“ clear as first Principles are puzzled; and
“ Terms which should be steadily used, are
“ ambiguous: But, notwithstanding all this
“ Address and Skill, the Point of getting rid
“ of ab cannot be obtained by legitimate
“ Reasoning. If a Man, by Methods not
“ geometrical or demonstrative, shall have sa-
“ tisfied himself of the Usefulness of certain
“ Rules; which he afterwards shall propose
“ to his Disciples for undoubted Truths;
“ which he undertakes to demonstrate in a
“ subtle Manner, and by the Help of nice
“ and intricate Notions; it is not hard to con-
“ ceive, that such his Disciples may, to save
“ themselves the Trouble of thinking, be in-
“ clined to confound the Usefulness of a Rule
“ with the Certainty of a Truth, and accept
“ the one for the other; especially if they are
“ Men accustomed rather to compute than to
“ think; earnest to go fast and far, than so-
“ licitous to set out warily, and see their Way
“ distinctly. “ The

78 *A preliminary Discourse.*

“ The Points or mere Limits of nascent
 “ Lines are undoubtedly equal, as having no
 “ more Magnitude one than another; a Li-
 “ mit as such being no Quantity. If by a
 “ Momentum you mean more than the very
 “ initial Limit, it must be either a finite
 “ Quantity or an Infinitesimal. But all finite
 “ Quantities are expressly excluded from the
 “ Notion of a Momentum. Therefore the
 “ Momentum must be an Infinitesimal. And
 “ indeed, though much Artifice has been
 “ employed to escape or avoid the Admission
 “ of Quantities infinitely small, yet it seems
 “ ineffectual. For ought I see, you can ad-
 “ mit no Quantity as a Medium between a
 “ finite Quantity and Nothing, without ad-
 “ mitting Infinitesimals. An Increment ge-
 “ nerated in a finite Particle of Time, is itself
 “ a finite Particle; and cannot therefore be a
 “ Momentum. You must therefore take an
 “ Infinitesimal Part of Time wherein to ge-
 “ nerate your Momentum. It is said the
 “ Magnitude of Moments is not considered:
 “ And yet these same Moments are supposed
 “ to be *divided* into Parts. This is not easy
 “ to conceive, no more than it is why we
 “ should take Quantities less than A and B in
 “ order to obtain the Increments of AB, of
 “ which proceeding it must be owned the fi-
 “ nal Cause or Motive is very obvious; but it
 “ is not so obvious or easy to explain a just
 “ and

“ and legitimate Reason for it, or shew it to
“ be Geometrical.”

Thus he goes on enforcing and redoubling his Objections in the liveliest Manner. But I must not follow him through the whole Pamphlet: I refer my Reader therefore to that for the rest, with which he cannot but be highly entertained, so much Life, Spirit and Genius appears in every Part of it.

XLVI. Two Persons entered the List in this Controversy against the Analyst; the former of which styles himself *Philaethes Cantabrigiensis*; the other was Mr. *Walton*, a Professor at *Dublin*. The Author of the Analyst answers them both in one Pamphlet, which he calls *A Defence of Free-thinking in Mathematics*. To this *Philaethes* made a Reply, so far as it concerned him; which was taken no Notice of by his Lordship: And there, as far as is come to my Knowledge, the Dispute ended.

XLVII. There is one Article in his Lordship's Answer, out of which I shall make a short Abstract, because it will place before my Reader, the Main of his Objections in one View: And shall then leave him, and shew how his Adversaries have considered this Subject.

Being wrought up to a small Degree of Warmth by his Antagonist *Philaethes*, to whom he is there addressing himself, he collects his whole Force, and levels all his Artillery

80 *A preliminary Discourse.*

tillery at Sir *Isaac* at once, in these Words :
 “ You, Sir, with the bright Eyes, be pleased
 “ to tell me, whether Sir *Isaac*’s Momentum
 “ be a *finite Quantity*, or an *Infiniteſimal*, or
 “ a mere *Limit*. If you ſay, a *finite Quan-*
 “ *tity*: Be pleaſed to reconcile this with what
 “ he ſaith in the Scholium of the ſecond
 “ Lemma of the firſt Section of the firſt Book
 “ of his Principles: *Cave intelligas quantitates*
 “ *magnitudine determinatas, ſed cogita ſemper*
 “ *diminuendas ſine Limite*. If you ſay, an
 “ *Infiniteſimal*; reconcile this with what is
 “ ſaid in the Introduction to his Quadratures:
 “ *Volui oſtendere quod in Methodo Fluxionum*
 “ *non opus ſit figuras infinite parvas in Geome-*
 “ *triam introducere*. If you ſhould ſay, it is
 “ a mere *Limit*, be pleaſed to reconcile this
 “ with what we find in the firſt Caſe of the
 “ ſecond Lemma in the ſecond Book of his
 “ Principles: *Ubi de lateribus A et B deerant*
 “ *Momentorum dimidia, &c.* where the Mo-
 “ ments are ſuppoſed to be *divided*.”

XLVIII. Both *Philaethes* and Mr. *Walton* explain very well what is meant by the Proportion, Moments or Differences bear to each other, but abſolutely decline giving any Account of the Things themſelves, any farther than that they repreſent them as Quantities infinitely diminiſhed; as Quantities in their naſcent and evaneſcent State. And tell us, that when they are rejected, it is becauſe they
 are

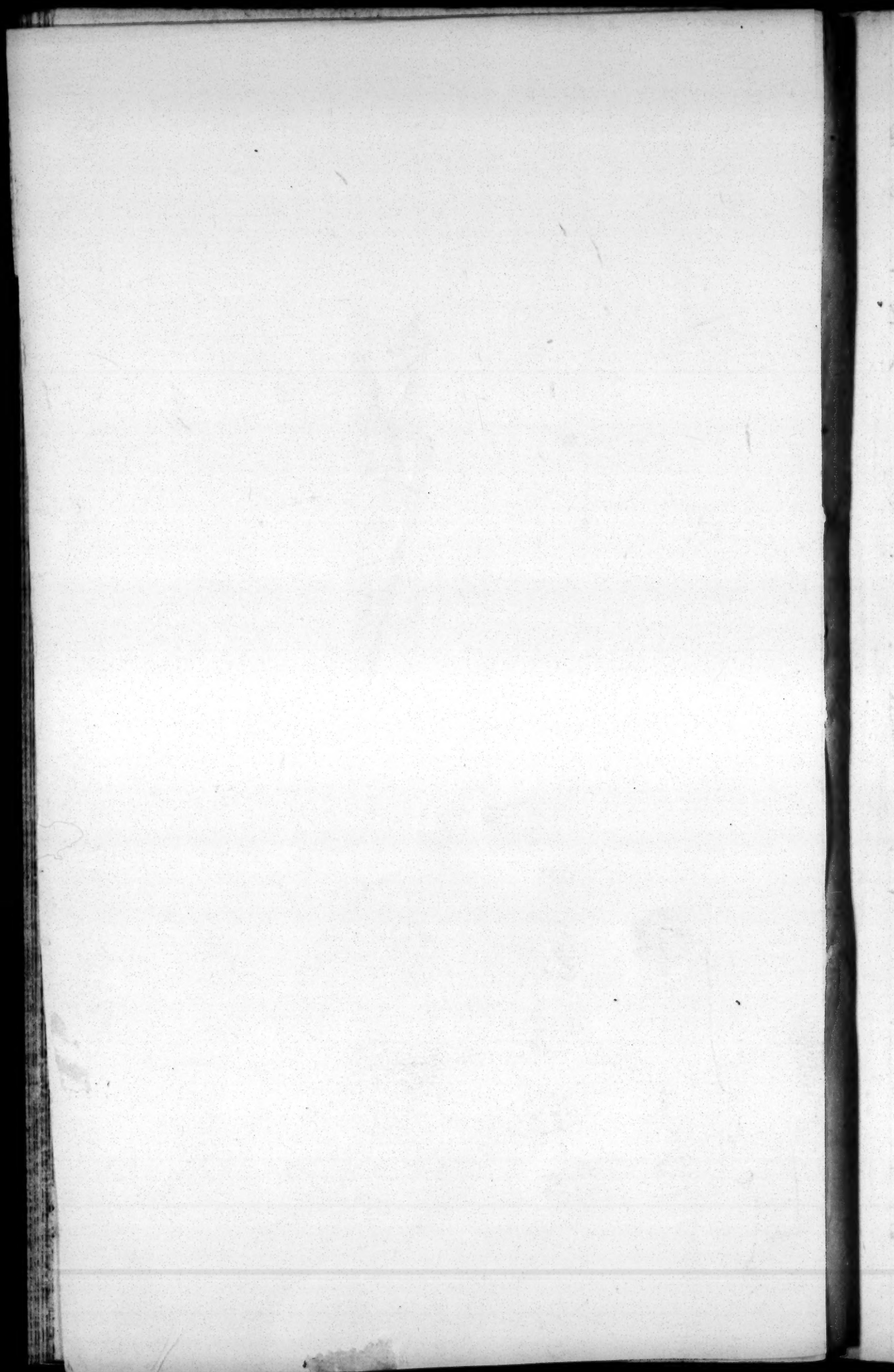


Fig. 1.

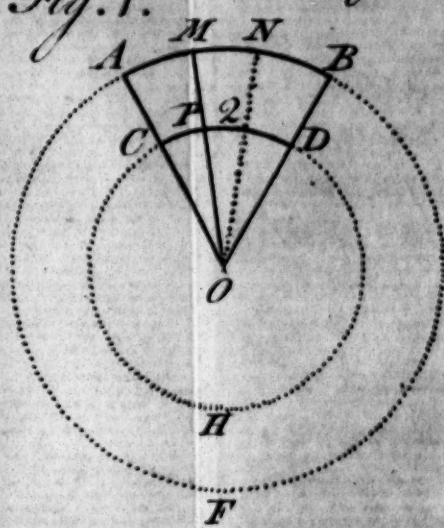


Fig. 2.

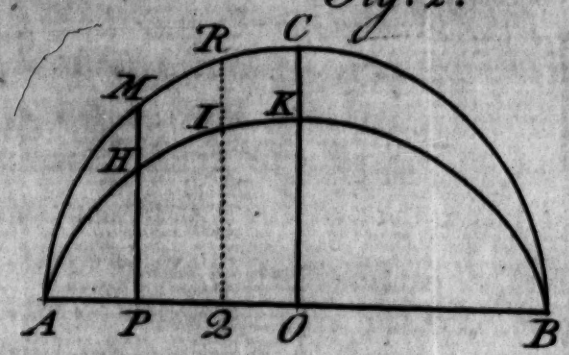


Fig. 3.

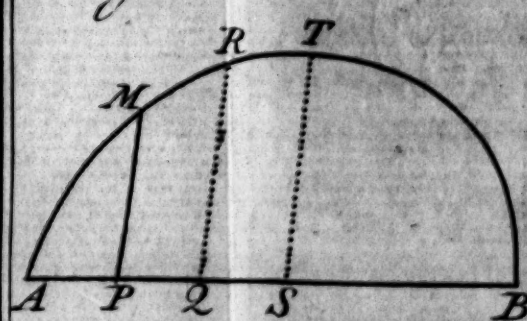


Fig. 4.

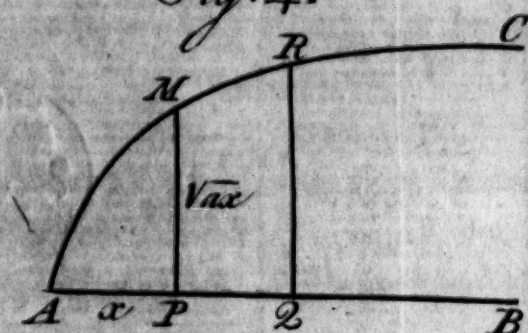


Fig. 5.

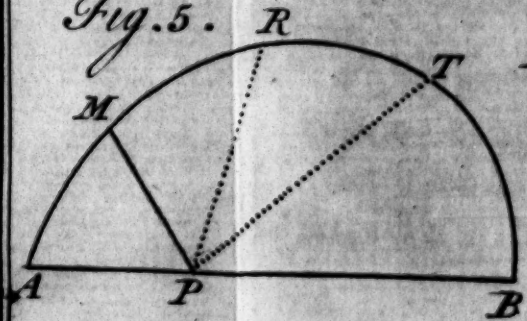
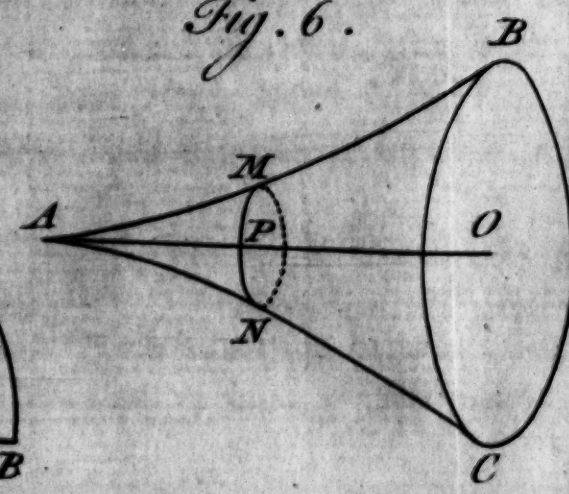
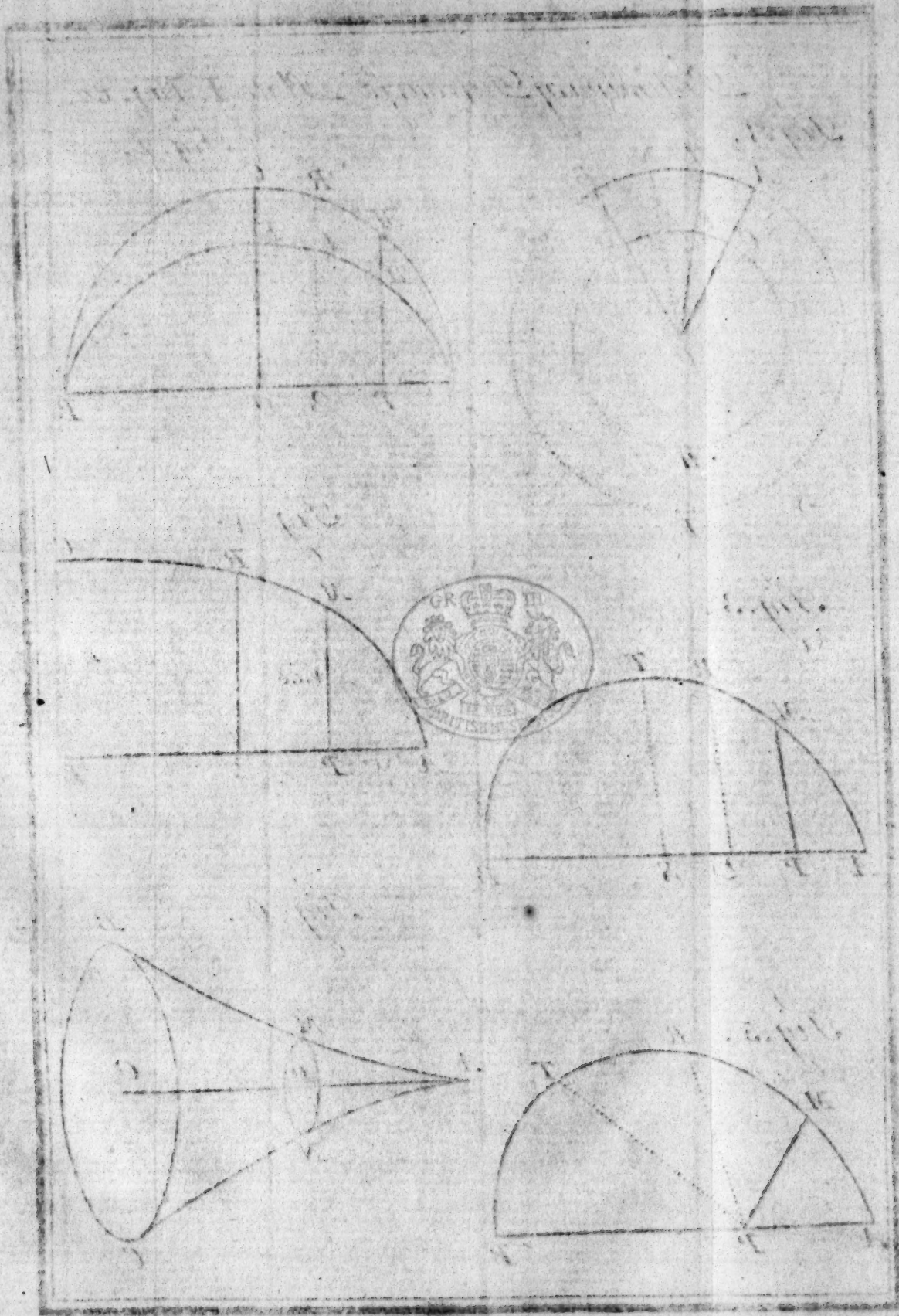
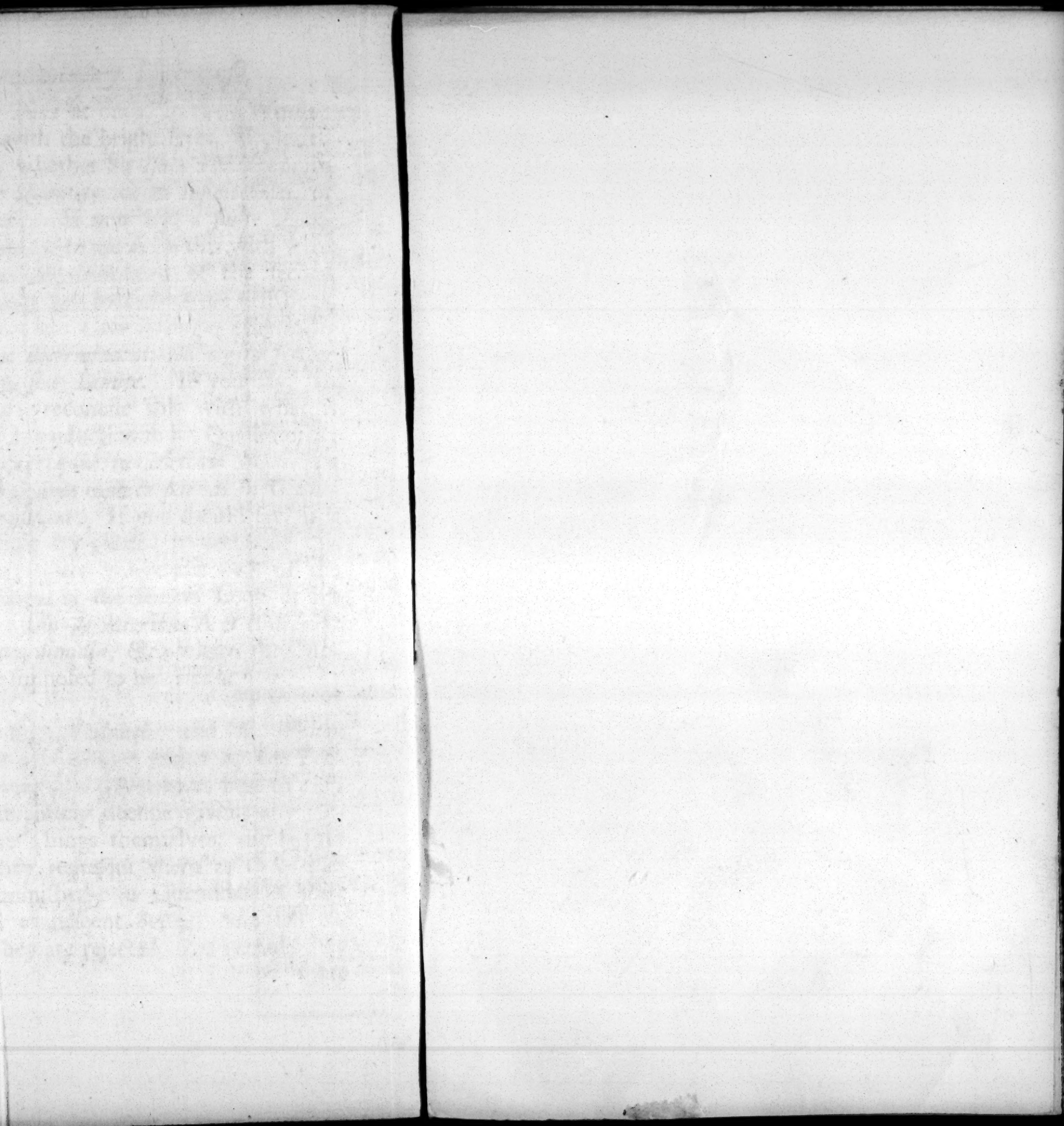
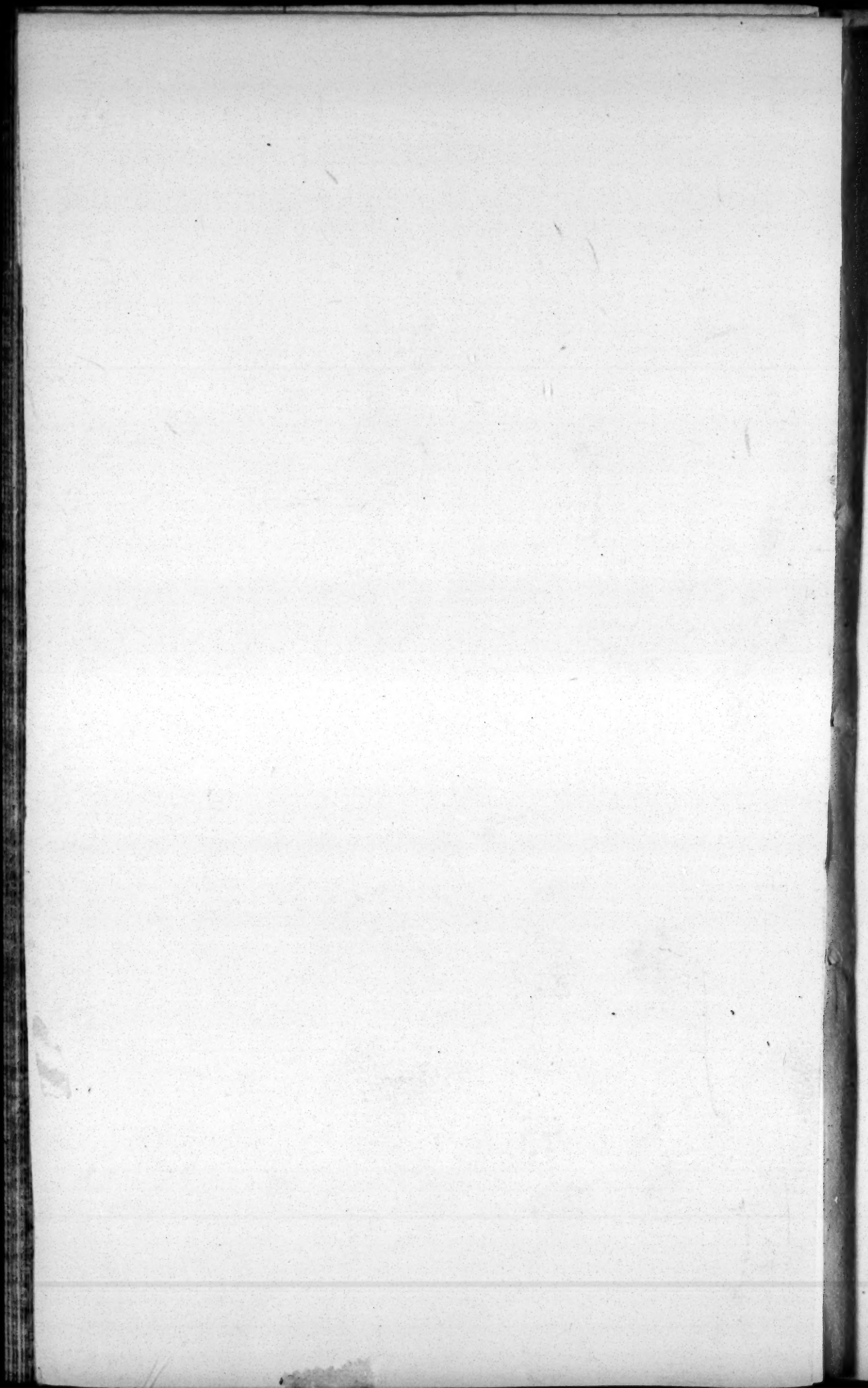


Fig. 6.









are infinitely small, and the like. They carry Things therefore no farther than Sir *Isaac* had done before. They leave them, as to the Objections made by the Analyst, exactly as they found them. I shall therefore trouble my Reader with no more Abstracts, than one very short one from each, to shew that I do them Justice.

Philaletbes ends his Account of the Proportion nascent and evanescent Increments bear to each other, in these Words: — “ It is
“ carefully to be attended to, that the Proportion here given, as the Proportion of the
“ evanescent Increments, is not their Proportion before they vanish. Nor is it their
“ Proportion after they have vanished. For
“ then they are become Nothing, and have
“ no Proportion. But it is their Proportion
“ at the Instant that they vanish, or the Proportion with which they vanish. I might
“ observe farther, that as the Increments do
“ not come to this Proportion before they
“ vanish, so neither do they vanish before
“ they come to this Proportion: But at one
“ and the same Instant of Time, they come
“ to this Proportion and vanish, they vanish
“ and come to this Proportion.” *

And when urged by the Analyst in his Reply, to say whether a Momentum or nascent Increment be a finite Quantity, or an Infinitesimal,

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* Second Letter, p. 30.

82 *A preliminary Discourse.*

tesimal, a mere Limit, or Nothing at all? puts him off in these Words: * — “ Would
 “ you have me tell you what a Moment is,
 “ or what the Magnitude of a Moment is? If
 “ the former; I tell you what Sir *Isaac New-*
 “ *ton* has told you before, a Moment is a
 “ momentaneous or nascent Increment pro-
 “ portionable to the Velocity of the flowing
 “ Quantity. If the latter; I have no Busi-
 “ ness at all to consider the Magnitude of a
 “ Moment. *Neque enim spectatur*, says Sir
 “ *Isaac Newton*, † *magnitudo Momentorum*,
 “ *sed prima nascentium Proportio*. I may tell
 “ you farther, that the Magnitude of a Mo-
 “ ment is Nothing fixed and determinate; is
 “ a Quantity perpetually fleeting and altering,
 “ till it vanishes into Nothing; in short, that
 “ it is utterly unassignable: *Dantur ultimæ*
 “ *quantitatum evanescentium rationes, non dan-*
 “ *tur ultimæ magnitudines.*” ‡

By Moments Mr. *Walton* tells us, || “ we
 “ may understand the nascent or evanescent
 “ Elements or Principles of finite Magnitudes,
 “ but not Particles of any determinate Size,
 “ or Increments actually generated; for all
 “ such are Quantities themselves generated of
 “ Moments.”

“ The Magnitudes of the momentaneous
 “ Increments or Decrements of Quantities are
 “ not

* Second Letter, p. 55.

† Princip. Lib. 2. Lemm. 2.

‡ Principia, Lib. 1. Sect. 1. Schol.

|| Page 6.

“ not regarded in the Method of Fluxions,
“ but their first and last Proportions only ;
“ that is, the Proportion with which they
“ begin or cease to exist : These are not their
“ Proportions immediately before or after
“ they begin or cease to exist, but the Pro-
“ portion with which they begin to exist or
“ with which they vanish. — Hence—to ob-
“ tain the last Ratio of syncronal Increments,
“ the Magnitude of those Increments must
“ be infinitely diminished. For their last
“ Ratio is the Ratio with which they vanish
“ or become Nothing by a constant Diminu-
“ tion, till they are infinitely diminished ; for
“ without an infinite Diminution they must
“ have finite or assignable Magnitudes, and
“ while they have finite Magnitudes they
“ cannot vanish.

“ The ultimate Ratios with which syncro-
“ nal Increments of Quantities vanish, are not
“ the Ratios of finite Increments, but Limits
“ which the Ratios of those Increments at-
“ tain by having their Magnitudes infinitely
“ diminished : The Proportion of Quantities
“ which grow less and less by Motion, and at
“ last cease to be, will in most Cases conti-
“ nually change, and become different in
“ every successive Diminution of the Quan-
“ tities themselves. And there are certain
“ determinate Limits to which all such Pro-
“ portions perpetually tend, and approach
“ nearer than by any assignable Difference,

84 *A preliminary Discourse.*

“ but never attain before the Quantities them-
 “ selves are infinitely diminished ; or till the
 “ Instant they evanesce and become Nothing.
 “ These Limits are the last Ratios with which
 “ such Quantities or their Increments vanish
 “ or cease to exist ; and they are the first Ra-
 “ tios with which Quantities, or the Incre-
 “ ments of Quantities, begin to arise or come
 “ into Being.”

XLIX. Hence it is easy to perceive, how far these Gentlemen are from giving an Account of those small Quantities satisfactory to the Analyst. In short, they give no other Idea of them, than that very one on which the Objections of the Analyst are wholly founded.

L. With regard to this Dispute then it must be owned, that these infinitely small Quantities, I mean, those nascent and evanescent Increments and Decrements of Sir Isaac Newton, and those Differences of Leibnitz, are not to be conceived by the Mind of Man ; and that no just and adequate Idea of them, as to Magnitude, can be formed. But then it is to be remembered on the other hand, that those Quantities are not Fluxions themselves, but only certain Measures or Representatives that have been made use of for Fluxions : That, as to the Fluxions themselves, there neither is, nor can be any Dispute about them, when rightly understood. A first Fluxion

Fluxion is the present Velocity wherewith a Quantity increases or diminishes, considered separately and apart from any Change or Alteration that Velocity may be undergoing. In this there is Nothing either infinitely great or infinitely little: Nothing obscure or unintelligible in the least. A second Fluxion is the Change in the Velocity a Quantity is at any Time increasing with, exclusive both of the Velocity itself, and also of any Alteration that Change may itself be undergoing. This necessarily takes place at every Instant of Time, while a Quantity increases faster and faster, or slower and slower; because in that Case there must be perpetually such a Change. Again, a third Fluxion is any Alteration that Change may at any Time be undergoing, exclusive not only of the Velocity the Quantity increases with, but also of the Change that Velocity may be undergoing at that Time; and also of any Change that Alteration itself may be undergoing. A fourth Fluxion is the Change that last mentioned Alteration may be undergoing, exclusive of all the former, and also of any Variation or Alteration that may be then happening to this very Change itself; and so on, as far as you please. These Things indeed elude our Senses; but they do not surpass the Understanding. There is nothing in them uncommon or unnatural in the least. They are happening daily in almost all the Motions, that fall under our Observation:

86 *A preliminary Discourse.*

And it is the Business of the Fluxionist to separate, and to express, and to consider them apart one from another.

LI. Those infinitely little Quantities, those Measures, I say, that have been made use of for Fluxions, it must be acknowledged, are, as to their Magnitudes, absolutely inconceivable; but, as to their Relations, they are most perfect and most compleat Representatives of Fluxions. For, as those of each Order are capable of comparative Magnitude among themselves, they are therefore capable of representing any Relations the Fluxions of that Order may bear to each other. They are therefore abundantly sufficient for Practice; because in Practice we have only to do with the Relations of Things; and the Relations of Things may be well understood, though our Ideas of the Things themselves be imperfect. And farther, as those of each successive Order are infinitely less than those of the foregoing one, as we shall see by and by, they do also most accurately and justly denote that infinite Inferiority and Subordination there is between the Fluxions of different Orders. But to wave all these Considerations for the present, and come nearer the Point.

LII. The Objections of the Analyst are distinguishable into two Parts. First, That infinitely small Quantities have been made use of for the Measures of Fluxions, or to represent them by, notwithstanding such Quantities,

ties, as to their Magnitudes, cannot be comprehended by the Mind: And, that such Quantities are in some Cases retained and made use of for a while, and afterwards, to use his own Expression, like Scaffolds to a Building, are rejected as of no Significancy.

LIII. In Answer to the first Part, it is to be owned, as I said before, that such Quantities have been used; and that they cannot be conceived by the Mind: But this is no Objection to the Method of Fluxions itself: Because finite Measures might have been made use of. From whence it follows, that the representing Fluxions by infinitely small Quantities was only casual and accidental, not of Necessity, but of Choice. And, in Answer to the second, it may be demonstrated, that those Quantities which are rejected in the fluxionary Method, are always such as ought by no means to be retained.

LIV. These two Points I shall endeavour to make out, in as few Words as possible, they being to be more forcibly demonstrated in the Book itself. In order to this, I must lay down the following fundamental Observation, *viz.*

LV. That when a Quantity increases with an uneven Pace, it increases neither faster nor slower at any one Point, or at any one Instant of Time, than if it increases at that Instant with an even Pace. Accordingly, when a Quantity is said to increase with an accelerated

88 *A preliminary Discourse.*

Motion, the Meaning is not, that it increases faster and faster at any one Instant of Time; it would be absurd to say it did; but only that it increases in one Instant quicker than it did the foregoing one. This may possibly appear plainer, if we take an Instance from a Body in Motion: I say then, that the Velocity of any Body is the same at any one Point, or at any one Time, whether the Body moves with an uniform, accelerated, or with a retarded Motion at that Point or Time. Thus, if a Body in Motion comes to a certain Point, suppose with ten Degrees of Velocity, it shall move over that Point with but ten Degrees of Velocity, though its Velocity be there accelerated; and it shall move over that Point with no less than ten Degrees of Velocity, though its Velocity be retarded in that Point. If the Motion be accelerated, it undergoes a *Change* indeed in the Point, and becomes greater afterwards; and if it be retarded, it becomes less, but not in the Point itself. This being allowed,

LVI. Let the right Lines AB and CD (see Fig. 13.) be parallel to each other; and let the Perpendicular PM , by moving parallel to itself from AC towards BD with an uniform and even Pace, generate the Area $APMC$ in the Manner we have so oft mentioned. Let PM be now supposed to have advanced as far as QI ; that is, let the Area $APMC$ be now supposed to have acquired the

the parallelogramic Increment PI ; then, whatever be the Breadth, MI , or PQ , of that Increment, that is, whatever be the Distance between the Lines PM and QI , if we suppose PQ to represent the Fluxion of the Base AP , that Increment PI shall represent the Fluxion of the Area CP .

For the describing Line PM being supposed to move with an even Pace, and the Lines AB and CD being parallel to each other, it is evident PM will continue of the same Length all the Way, and will therefore sweep over equal Areas in equal Times; and consequently the Area AM will increase during that Time with an even Pace, and the Line AP will also increase uniformly: But while AP shall gain PQ , AM will gain PI ; therefore, if the Velocity of PM , that is, the Fluxion of AP , be represented by PQ , the Fluxion of the Area AM shall be represented by the Parallelogram PI ; and consequently, if we call PQ , $\dot{x}$, $PM \times \dot{x}$ shall be the Value of this Fluxion.

Secondly, Let CMD be a curve Line, and situated with respect to AB , as in Fig. 14. and let the Area $APMC$ be described by the Line PM moving uniformly forwards, as before; and let the other Lines be drawn as represented in that Figure. Here PM , as it moves along towards BD , will lengthen continually, because the Distance between the Lines CMD and AB grows greater and greater;

greater; and therefore the Area $APMC$ will continually increase with an accelerated Motion; and when P is come to Q , and PM to QR , the Area will have acquired the Increment $PQRM$; but its Fluxion will be no greater, than if PM had all the Time continued invariable. For, by the fundamental Observation above laid down, though the Area increases faster and faster, yet when it is $APMC$ (which observe is but one single Instant of Time) it increases no faster there, than if it increased there with an uniform Pace, that is, no faster than it would have done, had MR been coincident with MI , or had PM continued of the same Length all the Way; and therefore, by what was shewn in the last Article, if PQ be supposed to represent the Velocity PM moves with, or the Fluxion of the Base AP , as before, the parallelogramic Increment PI shall here also represent the Fluxion of the Area $APMC$. And therefore if we call PQ , $\dot{x}$, $PM \times \dot{x}$ shall be the Value of this Fluxion also.

Again, let the Lines AB and CD be situated, as in Fig. 15. and let the other Lines be drawn, as they are there represented. Here the Line PM shall grow shorter and shorter, as it moves along; the Area $APMC$ shall flow slower and slower; and when PM is come to QR , the Increment of the Area shall be no more than $MPQR$. But, for Reasons altogether similar to those already given,

given, if PQ be supposed to represent the Fluxion of AP , not the Increment $PQRM$, but the Parallelogram $PQIM$, or $PM \times \dot{x}$ shall here also represent, or measure the Fluxion of the Area $APMC$.

LVII. From whence the following Observations naturally arise, *viz.*

First, that in all these Cases the Fluxion of the Area may be measured or represented by a *finite Quantity*, *viz.* by the finite Parallelogram PI ; and therefore, that the representing Fluxions, at least in these Cases (and we shall demonstrate for all in general in another Place) by infinitely small Quantities, was casual and accidental, and not essential to the Method.

Secondly, That when the Area flows with an accelerated Motion, while the Base flows uniformly, as was the Case, in Fig. 14. the total Increment that is produced, or that arises while the Area from AM becomes AR , exceeds the Parallelogram PI , which measures the Fluxion of the Area AM , by the Quantity MIR ; and consequently, that in order to find, from the total Increment PR , a Quantity that shall represent the Fluxion of the Area, that curvilinear Space MIR ought to be rejected therefrom. And

Thirdly, That when the Area flows with a retarded Motion, while the Base flows uniformly, as the Area AM , in Fig. 15. the Increment that arises is less than the Parallelogram PI by the curvilinear Space MRI ,

to

92 *A preliminary Discourse.*

to find therefore that Parallelogram or the Fluxion of the Area, from that Increment, such curvilinear Space must be added to it.

Fourthly, That, since the Parallelogram PI , in Fig. 13. is what arises in consequence of the Fluxion of the Area AM , when such Fluxion is supposed to be continued on without Alteration or Variation, it is obvious, that the Parallelograms PI , in Fig. 14, & 15. are what would have arisen in consequence of the Fluxions of the Areas in these Figures, had these Fluxions suffered no Alteration; and consequently that the curvilinear Space MRI , in Fig. 14. is what arises in consequence only of the Acceleration wherewith the Area flows; and that the like curvilinear Space MRI , in Fig. 15. is what is lost in consequence only of the Retardation wherewith the Area flows. And therefore that, in both Cases, such curvilinear Space is to be ascribed, and placed, to the Account of the second, and higher Fluxions of the Area, if such there be, and not to the first. From whence it clearly follows, that, in computing the first Fluxion of the Area, and expressing it separately, and apart from its other Fluxions, the deducting or throwing out such curvilinear Space from the total Increment PR in the one Case, and adding it thereto in the other, is a most just and reasonable Method of proceeding.

LVIII. Now,

LVIII. Now, since it might be difficult, in some Instances, to discover how much of the total Increment is to be taken from it in one Case, or how much is to be added to it in the other, to find the true Measure of the Fluxion of the Area; and, since the Line PQ or x may be taken of any Length, the great Author of this Method, directs that we conceive the Line QR to return back to PM , whence it is supposed to have set out; in which Case PQ or x will become infinitely short, and the Increment PR will vanish; but at the same Time that Increment vanishes, the Line RI will vanish also; because QR will then become equal to PM , and consequently to QI : So that the total Increment PR , and the parallelogramic Increment PI will then be one and the same Thing. The supposing therefore the Line PQ , or x , infinitely short, and the Increment PR in its vanishing State (or, which is the same Thing, in its nascent or rising State, if you suppose QR at PM , and departing from it towards QR); I say, the supposing these Things is only a most expeditious and ready Method of getting rid of the curvilinear Space MRI , when it ought to be rejected, and of supplying it in Cases where it ought to be added. This the Analyst contends is an illegitimate and unreasonable Method; whereas it is rather perhaps an Instance of the greatest human

man Penetration and Wisdom the World ever saw.

Hence the particular Objections of the Analyst admit of an easy and ready Answer.

LIX. The Objection which relates to the Method of establishing the Rule for finding the Fluxion of any Power, is this (see above, Article 44.) viz. that the Supposition which is made at the Beginning of the Process, and carried on to a certain Point, is then shifted for one that is contrary to it, and the remaining Part of the Process carried on upon the latter. That in order to get rid of the Terms

$$\frac{nn - n}{2} oo x^{n-2} +, \&c. \text{ (in the Quantity}$$

$$no x^{n-1} + \frac{nn - n}{2} oo x^{n-2} +, \&c.) \text{ the Let-}$$

ter *o*, which was at first supposed to represent a real Increment, is afterwards in the same Process, considered as no Increment or Nothing: Which, he insists upon it, is an illegitimate Manner of proceeding, and what would not be allowed of in other Sciences. But it may be shewn, without shifting the

Hypothesis, that the Quantities $\frac{nn - n}{2} oo x^{n-2} +,$

&c. are generated, or do arise, in consequence of the Acceleration wherewith the Power of *x* flows, when *x* itself flows uniformly; and consequently that they arise from the second and higher Fluxions of that Power; and that therefore,

therefore, when the first Fluxion of that Power is only inquired after, whether the Quantity o be supposed to continue a finite Increment or not, they are to be left out and rejected, as appertaining to another Account. To demonstrate this for any Power in general would be too much for this Place; I shall therefore now shew it for the Cube or third Power only. The Increment of x raised to the third Power, when the Root x was supposed to be augmented by the finite Quantity o , was (as observed above, Art. 43.) $3xxo + 3xoo + ooo$. I am to shew therefore, that the first of these three Terms only is to be taken for the Measure of the Fluxion of xxx ; and consequently that the other two are redundant. In order to this, let $APMC$, in Fig. 16. represent a Cube whose Side is x , and let the Cube be supposed to be augmented in each Dimension by a Line, as PQ , which we will signify by the Letter o . Then while the Cube increases, every Face of it, as it removes from its Place, will continually grow larger and larger, because the larger the Cube, the larger every Face of it will be; When the Face PM therefore is come to QI , it will be larger than QI or PM ; but for the same Reasons that in estimating the first Fluxion of the Area AM , in Fig. 14. we took the Measure of that Fluxion such, as might have been generated by the Motion of PM from P to Q without increasing itself, *viz.* the Parallelogram

gram PI ; so now, for the very same Reasons, we must take the Measure for the Increase of the Cube in this Dimension, such, as might be generated by the Motion of the Plane PM from P to Q , without undergoing any Alteration in itself; that is, we are to take the Parallelepiped PI for that Measure. Since then (each Side of the Cube being x) the Plane, Area or Face PM is xx , that Parallelepiped PI will be xxx ; this therefore is the Measure of the Rate the Cube will increase at in that Dimension; but it is evident, it will increase as much in each of its other two Dimensions; from whence it follows, that its whole Flux will be thrice that Quantity, viz. $3xxx$, which is the first of the three Terms of the Increment $3xxx + 3x^2\dot{x} + x^3\ddot{x}$, the remaining two therefore, viz. $3x^2\dot{x}$ and $x^3\ddot{x}$, are redundant: They are therefore what arises in Consequence of the Acceleration wherewith the Cube increases, that is, from the second and third Fluxions thereof, and are to be placed to their Account.

Another Difficulty the Analyst insists upon, is that of getting rid of the Quantity ab in the Increment $aB + bA + ab$ (see above, Art. 45.) but this is of the same Kind with the former; that Expression ab being that Part of the Increment only which arises from the Acceleration with which the Rectangle AB flows, and not from the Rate it flows at, and therefore

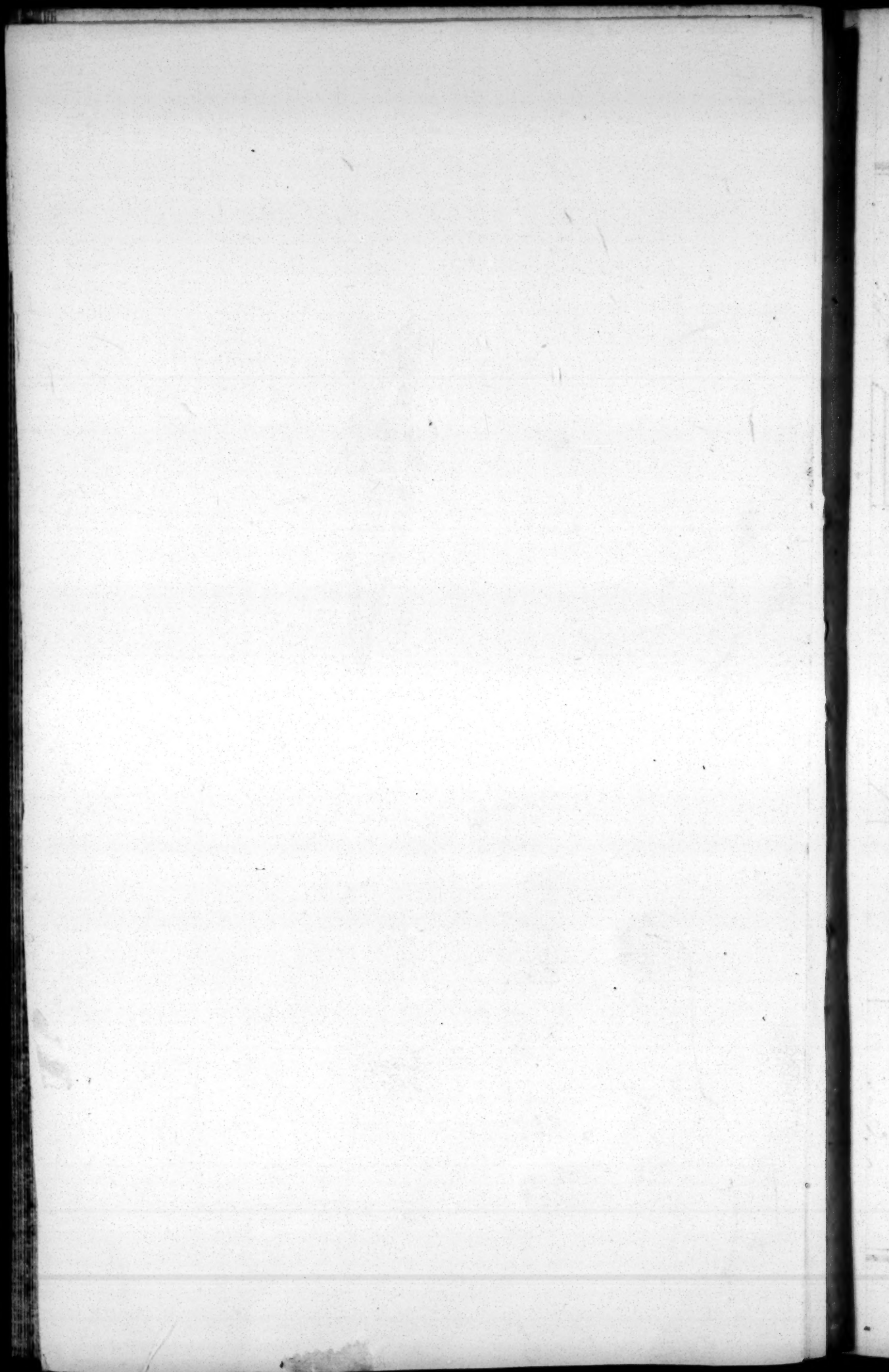


Fig. 7.

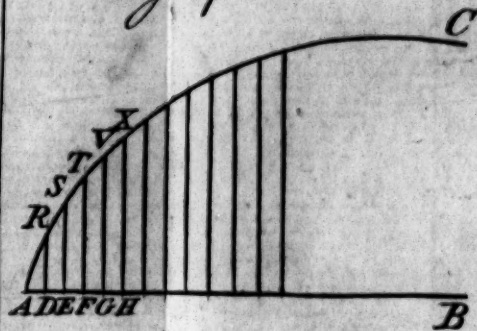


Fig. 8.

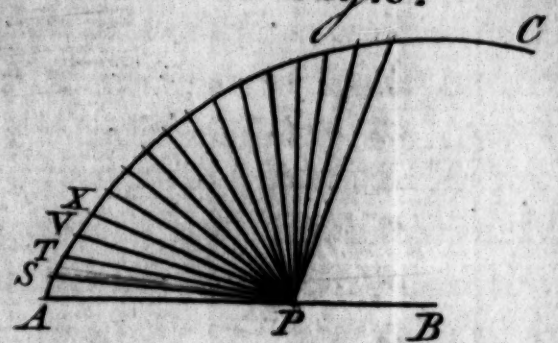


Fig. 9.

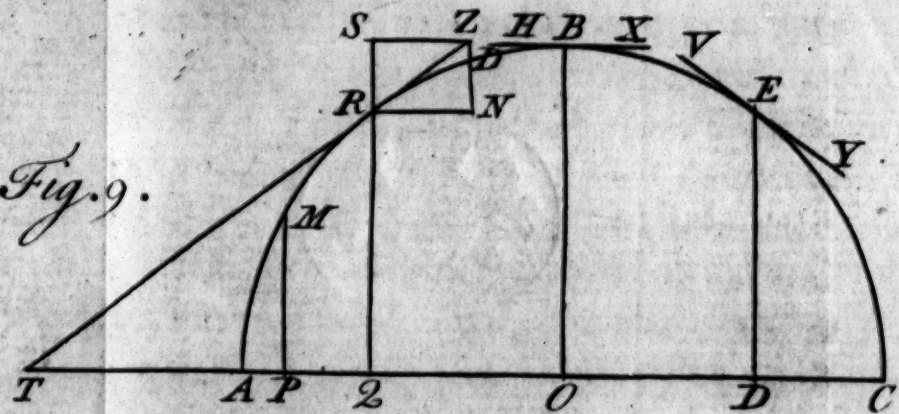


Fig. 10.

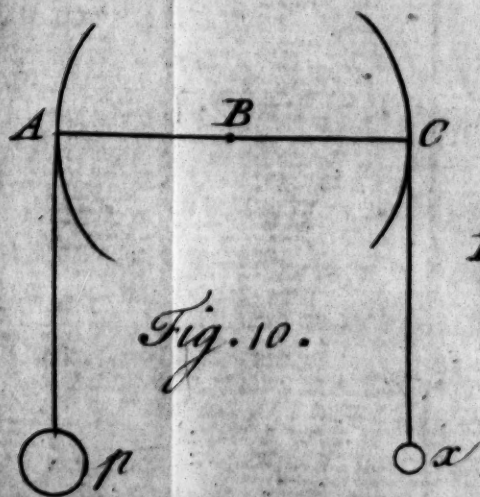
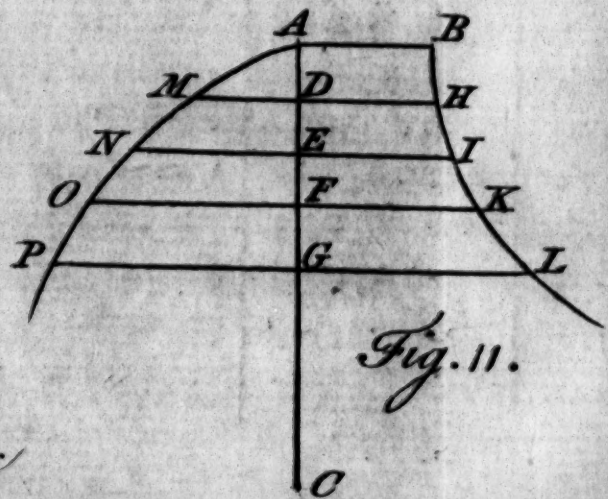
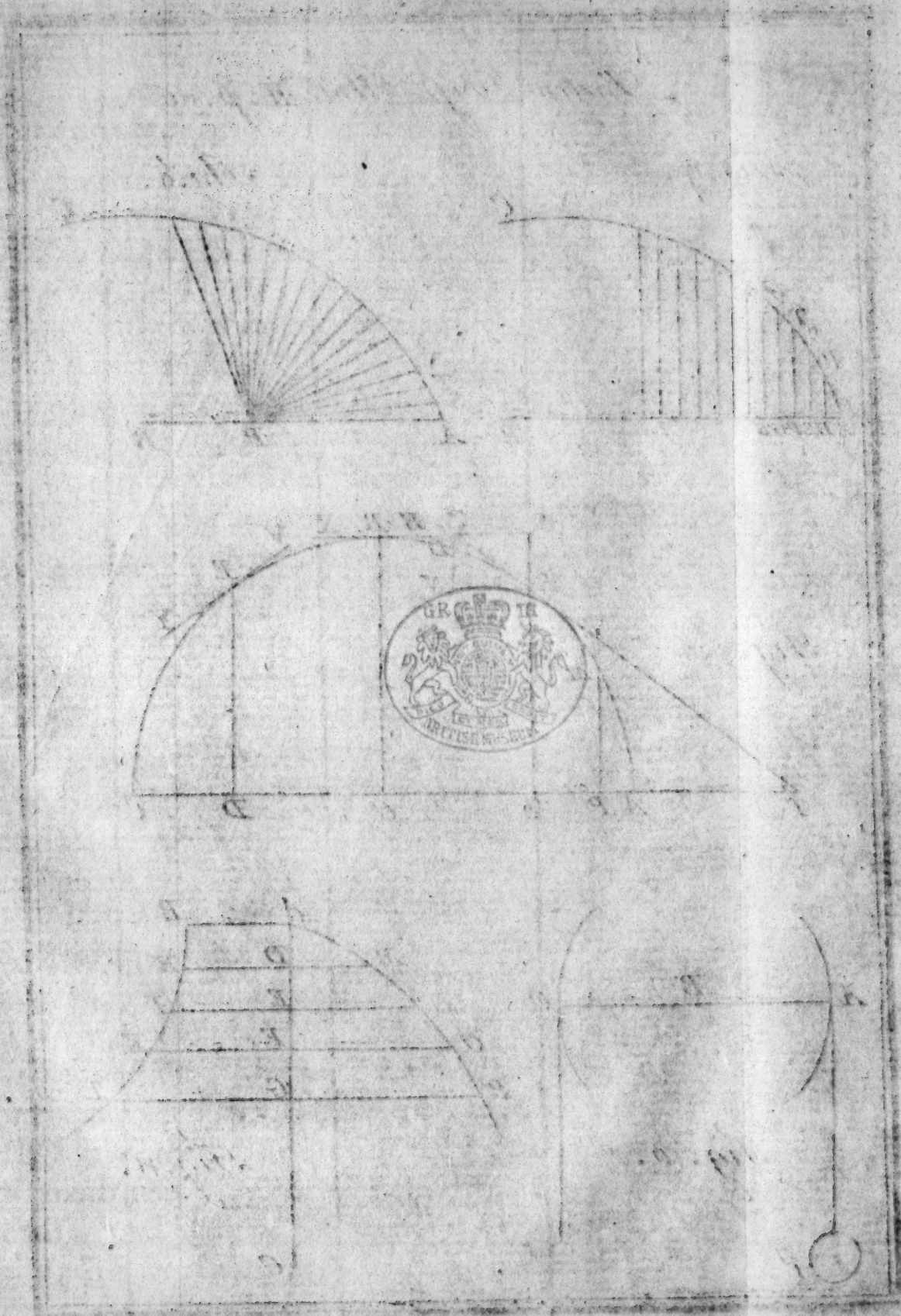
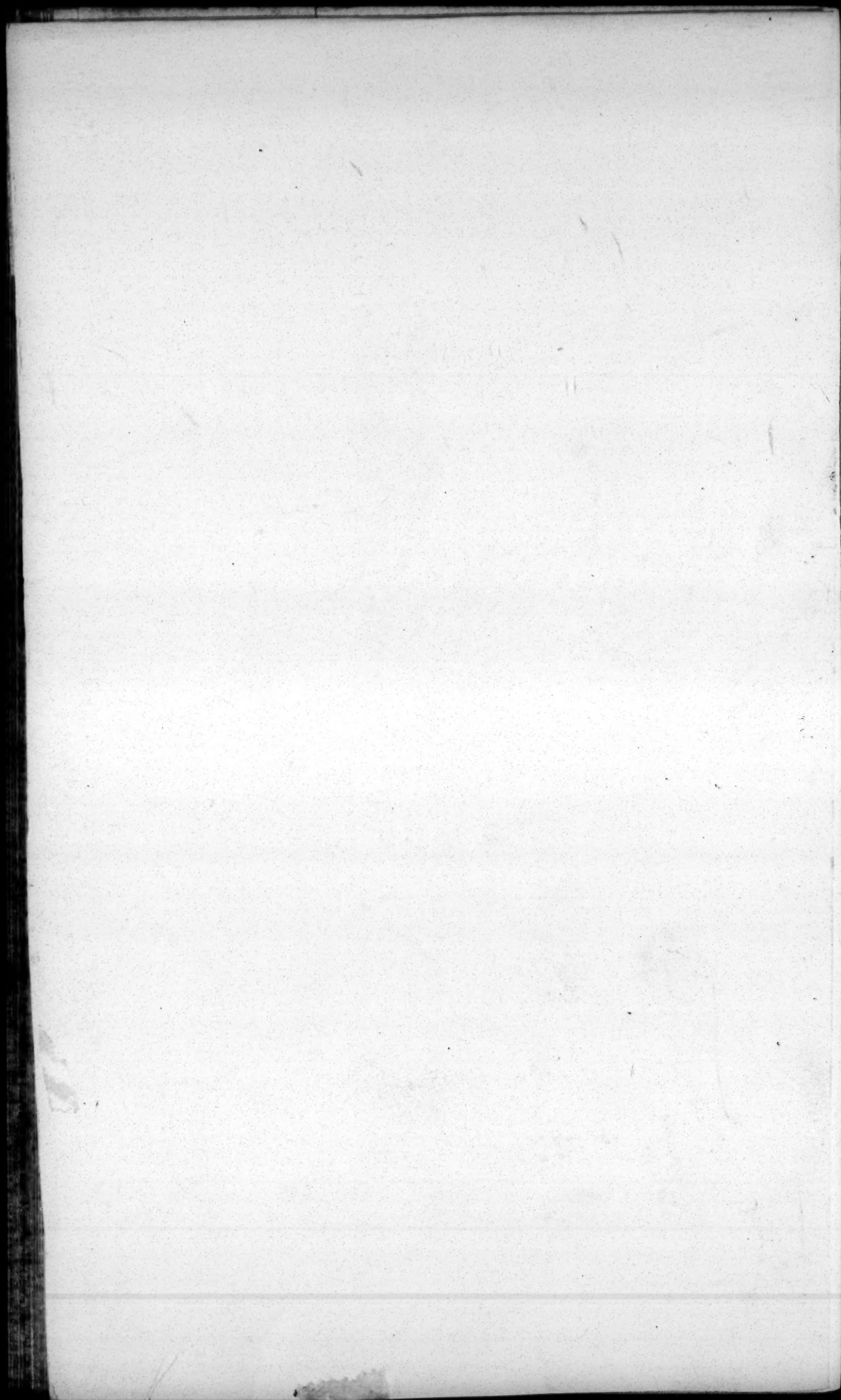


Fig. 11.





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therefore in computing that Rate, it is properly and duly rejected.

Another Objection is, Sir *Isaac Newton* has halved one of his Moments — But since those Moments are not essentially and of Necessity infinitely small, it is clear, that the taking, or considering them so, must not be supposed to divest them of that Divisibility they would otherwise have. But this will be spoke to in the next Article.

LX. To recapitulate then, and in some Measure further to illustrate the Nature of these Sort of Quantities, and their Relations. Let the Area *APMC*, in Fig. 14. be supposed to be described by the Motion of the Line *QR* from *AC* towards *BD*; and let it be supposed, that the Line *QR* is now actually upon, and coincident with the Line *PM*; call the then Distance between them $\dot{x}$; multiply *PM* by that Distance, and you will have $PM \times \dot{x}$ an infinitely small Quantity indeed, because $\dot{x}$ will be Nothing at all: It will be absolutely indivisible in the Dimension $\dot{x}$, and will by no means express the Fluxion of the Area *AM*; because there will be Nothing therein that respects the Velocity the Line *QR* moves with in describing that Area. But though there be no actual Distance between those Lines, when they are coincident with each other, yet as one of them, *viz.* *QR* is in Motion, the Distance between them is

H

then

then from positive on one Side the Line PM becoming negative on the other, or the contrary; and this with a certain Degree of Quickness, according as that Line is supposed to move fast or slow: Let $\dot{x}$ represent that Degree of Quickness: Then multiply PM by $\dot{x}$, and the Product, $PM \times \dot{x}$, shall represent the Degree of Quickness wherewith the Increment from positive becomes negative, or from negative becomes positive: It shall now respect the Velocity the Area AM flows with, or the true Fluxion of the Area; and therefore, as that Degree of Quickness represented by $\dot{x}$, may be great or small, $PM \times \dot{x}$ may be so too; it will therefore now be capable of Relation and consequently of Divisibility.

It appears therefore, that when $PM \times \dot{x}$ is supposed to denote, or measure the Fluxion of the Area, the Quantity $\dot{x}$ must necessarily be supposed, and considered, as the Measure of the Velocity the Line PM moves with, which Velocity being always finite, it is evident the Quantity $PM \times \dot{x}$ is always naturally and constitutionally a finite Quantity; because, as we have just seen, it is otherwise by no means capable of representing that Fluxion. When therefore for Purposes of Conveniency, Expedition or Brevity, it is said to be in its nascent or evanescent State, to be just coming into Being, or just departing out of it; to be less than any Quantity that can be

be given ; to be infinitely small, or infinitely diminished ; to be an Element of some finite Magnitude ; or the like ; it only implies, that QR is just then moving over PM with the finite Velocity $\dot{x}$, that $PM \times \dot{x}$, considered as an Area, is a Quantity changing from positive to negative, and therefore as yet infinitely small, or infinitely smaller than any Quantity in which such Change is actually made. But when $\dot{x}$ is considered as a Velocity, $PM \times \dot{x}$ is a Quantity actually finite : But it is essential, that it should be so considered ; because otherwise, as observed above, it can by no means represent, or express, the Fluxion of that Area. It is therefore essentially and constitutionally a finite Quantity, and therefore essentially *divisible*. When it is considered as an infinitely small Quantity, it is only indirectly, nominally, and by Supposition, such, and purely for the Sake of divesting it of the curvilinear Space $MR I$, or of supplying it with that Space, without the Ceremony of a long Deduction.

LXI. I explain this a little further. I observed, near the Beginning of this Discourse, that, in the Resolution of some Problems, as the finding the Areas of Figures ; drawing Tangents to Curves, &c. the first Fluxions of Quantities are alone sufficient ; and that in other Problems, second Fluxions are made Use of : In like manner, there are

others, where third Fluxions are concerned, &c. and accordingly I took Notice afterwards, that it is the Business of the Fluxionist to separate and part these one from another, in order to make use of such, and such only, as the Case may require. Imagine then the Line $CMRD$ in Fig. 14. to be a right Line, then will MRI be a right-lined Triangle, and (for Reasons similar to those already given) the parallelogramic Increment PI will measure the first Fluxion of the Figure AM ; and for Reasons we shall give afterwards, twice the Triangle MRI shall measure the second Fluxion of that Figure. Suppose that I have Occasion for the first Fluxion of this Figure only: To obtain this from the total Increment PR , it is evident, I must reject the triangular Space MRI , as above observed. But suppose, that from the the total Increment PR , I would obtain the second Fluxion of the Figure; here it is evident, I must reject PI that measures the first Fluxion, and reserve and make use of the triangular Space I rejected before. Suppose QR to return back to PM , so that PR may be what they call an infinitely small Quantity; the Case is still the same; I must reject PI to get MRI ; so that here I reject an infinitely small Quantity, and retain one that is infinitely smaller: The Reason of this can never be, because the former is infinitely less than

A preliminary Discourse. 101

than the latter; for, in their own Way of speaking and expressing it, it is infinitely larger. It appears therefore that the true Reason, why Quantities are rejected in Cases of this Sort, is not because they are infinitely smaller than others. The true Reason is, that they are not wanted, as having no Part or Share in the Quantity required. Those Mathematicians who lay too much Stress upon the Rejection of Quantities, because they are infinitely smaller than others, will do well to consider this. Hence it appears also, that the Argument, whereby the first Fluxions of Quantities have been usually deduced, is certainly not strictly and truly *logical*. It is no other than an Argument *ad Ignorantiam*, though one of the most subtle and most ingenious of that Sort, that ever was devised, or thought of.

LXH. There is one Thing more, which it may be proper to take Notice of, relative to this Doctrine, and then I think I shall have sufficiently explained the Nature and Ground-work of it, *viz.* That it may seem a little odd to Beginners, that in estimating the Magnitude of a Quantity, in the fluxionary Method, we should not make use of its whole Increment (as PR), but only so much of it, (as PI) as measures its first Fluxion, when it is clear, that the Magnitude of a Quantity

can be no less than all it shall have acquired by increasing: or, that we should make use of only the first Fluxion of the Quantity, when perhaps second, third, fourth, fifth, sixth, &c. Fluxions *ad infinitum*, may actually have been all concerned in the *Genesis* of that Quantity.

In answer to this, it is to be remembered, that when, in estimating the Magnitude of Quantities, we infer, that two Quantities are equal to each other; it is only when their first Fluxions, as I largely explained it before, have been found to be equal from first to last, or during the whole Time of the Increase. But two Quantities cannot have their first Fluxions the same from first to last, or for any Continuation of Time, but their second, third, fourth, &c. Fluxions, if they have such, must necessarily be equal to each other during that Time. For the second, third, fourth, &c. Fluxions of any Quantity being, as observed above, the Changes and Subchanges the Rate it increases at, undergoes; it follows, that, if the first Fluxions of two Quantities be *constantly* the same, all the other Fluxions of those Quantities must be the same in each of them. The Equality of the second, third, &c. Fluxions is virtually implied in the continued Equality of the first. Thus, if you *accompany* another Person, you must stop when he stops, and alter your Pace as he does. The sole Circumstance

cumstance of *accompanying*, or having equal Paces of the first Order, during any Continuance of Time, implies, that those of every other Order, are the same also during that Time. When therefore, in computing the Magnitude of a Quantity, we suppose some one Term, or Dimension, in an algebraical (or other) Expression, to increase along with some one Dimension of the Quantity whose Magnitude is required, and find, by Computation, that the Quantities themselves, *viz.* the algebraical, (or other) Expression and the Quantity sought, must have had the same first Fluxions, that is, must have accompanied each other in increasing all the Time, we justly conclude, that when they are equal to each other in their above-mentioned Dimensions they will also be equal in themselves: For though their first Fluxions only have been taken into the Account, yet, as those first Fluxions have continued equal in each Quantity all the Way, it is clear, that all the Changes and Subchanges those Fluxions may have undergone, must have been the same also in each: That is, as the first Fluxion of the one has been the same all the Way with the first of the other; the second also of the one must have been the same with the second of the other; the third of the one, the same with the third of the other; and so on. The Quantities themselves therefore must have increased during the whole Time in the same manner in all Respects, and

104 *A preliminary Discourse.*

therefore must be equal to each other, provided they were not unequal, when the Increase began.

It may not be amiss to take Notice also in this Place, though I mentioned it before (Article 9.) that by the Line PM, which is one Dimension in the Fluxions of the several Areas in Fig. 13. 14. and 15. must by no means be understood the perpendicular Distance between the Lines CD and AB at some determinate Point, as P, only; but an algebraical Theorem or Expression denoting that Distance, measure it where you will: For otherwise, as is evident, the Fluxions of different Quantities, as, for instance, the Fluxions of the different Areas in those Figures, might each be expressed the same Way; from whence the utmost Confusion would arise. The Distance of the Line PM from the Point A, is to be denoted by some general Character, as x , and its Value, or Magnitude, to be computed from the Nature and Properties of the Figure, or Quantity concerned; by which means, though the Line PM itself may be the same in different Cases, it shall come out differently expressed, and consequently the Parallelogram PI, of which it is always one Dimension, shall be also expressed in a different Manner. In short, it shall always be so expressed, as equally to relate to the Area AM, or the Quantity concerned, under all Magnitudes;
and

and hence it is, that, as I have sometimes expressed it, it shall measure the Fluxion of that Quantity *from first to last*.

LXIII. It was observed (Article 51.) that the Increments of Quantities, when considered as infinitely small, are, as to their Relations, proper Representatives of the first Fluxions of those Quantities; so, in like manner, are those which are deemed infinitely small, and those which are infinitely less than them, and so on, that is, those of the second, third, &c. Order, most apt and proper Measures of the Fluxions of each corresponding Order: And not only so, but they do also at the same Time naturally represent and express that Subordination and infinite Distance there is between the Fluxions of any one Order, and of that which immediately follows it. Which Thing I now proceed to explain.

LXIV. If we have due Regard to the Definitions of Fluxions of various Orders, *viz.* of First, Second, Third, &c. as I just now laid them down (Article 50.) we shall readily perceive, that the Fluxions of each Order are essentially different from those of the foregoing one, as the first Fluxion of every Quantity is from the Quantity itself. The first Fluxion of any Quantity with Respect to the Quantity itself, is no more than an Affection thereof, it is but the Velocity wherewith the Quantity increases;

increases; there is therefore an essential, that is, in mathematical Language, an infinite Difference between them. Again, a second Fluxion is but a Change or Alteration of Increase in the First: But 10000 Changes of Increase will not constitute any one Rate of Increase whatever; a Change of Increase is therefore in the Order of Things infinitely below the Increase itself. A third Fluxion is a Change or Alteration in the former, and is therefore also essentially different from that: It is in the Order of Things infinitely below it; and so on. Thus, there is an infinite Inferiority and Subordination between the Fluxions of any one Order, and of that which immediately preceeds it. This infinite Gradation is most beautifully represented in the very Nature and Constitution of those infinitely little Quantities we are now speaking of, when considered as they respect one another, and the Quantity whose Fluxions they represent.

To give an Idea of this; let $\dot{x}$ be the Increment of the Base AP, in Fig. 14. it shall then, as we have seen, enter the Expression for the Fluxion of the Area AM, and be a Factor therein, as in the Expression $PM \times \dot{x}$; that Fluxion therefore, if the Increment $\dot{x}$ be considered as infinitely small, will be infinitely smaller than the Quantity whose Fluxion it is, that is, than the Area AM; and therefore will aptly represent and express that infinite

nite Subordination and Distance there is between the Quantity and its first Fluxion. And, as we shall see afterwards, I mean in the Book itself, the Square of x , or x^2 , shall be a Factor in the second Fluxion of that Quantity; and will imply, that this Fluxion is x times less, that is, infinitely than the former: Again x times x times x , or x^3 shall enter the Third, and be a Factor there; which will imply, that this Fluxion is infinitely inferior to the former; and so on. And consequently when considered in this manner, they do most aptly shadow out, and express, that infinite Inferiority and Subordination above-mentioned.

Thus I have thrown together all I have to say by way of Explication of the Nature and Peculiarity of this Method; and shall therefore now proceed to the Treatise itself, for which this preliminary Discourse is intended as an Introduction; in which Treatise I propose to demonstrate the Rules of the Method both with and without the Supposition of infinitely small Quantities, or Quantities infinitely diminished, that the Reader himself may judge to which of the two different Ways the Preference ought to be given.

LXV. The Method of Fluxions was at first liable to another Objection, not taken Notice of by the Analyst, *viz.* that the Rules of it were

were established on the *Binomial Theorem*; the Truth of which Theorem was at that Time taken for granted; and was not of a long Time after demonstrated without the Help of Fluxions. But that Theorem has since been demonstrated without Fluxions: and farther, that incomparable Mathematician *Mac Laurin*, in his Treatise above mentioned, has demonstrated the Rules of the Fluxionary Method without that Theorem. The Doctrine of Fluxions is therefore now sufficiently guarded on that Side. No more therefore need be said of it, as to that Particular.

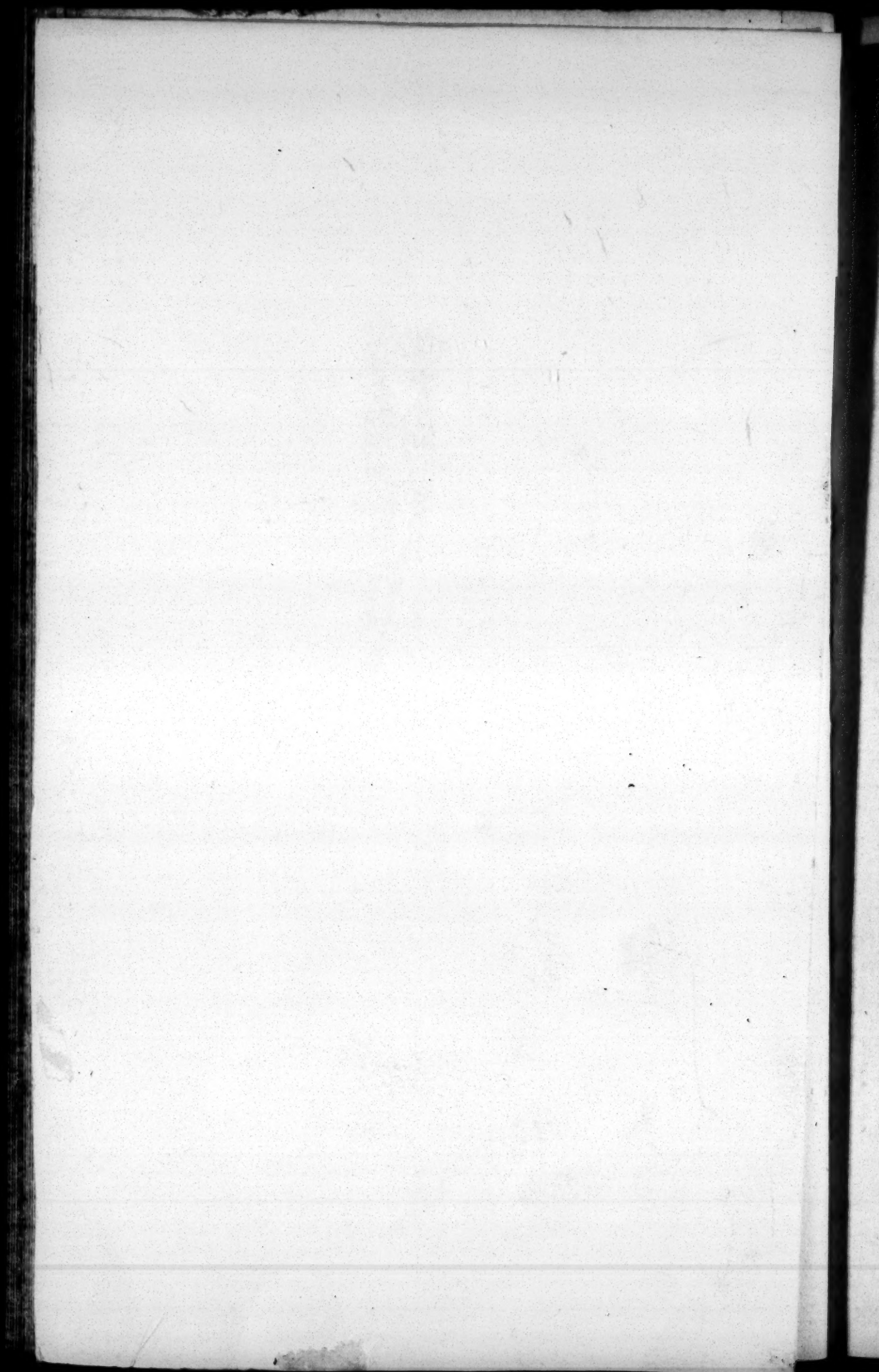


F I N I S.

E R R A T A.

Page 72, Line 14, for x_s read x'' .

Page 85, Line 25, for *may* read *any*.



Prelim. Disc.

Plate III. p. 108.

Fig. 12.

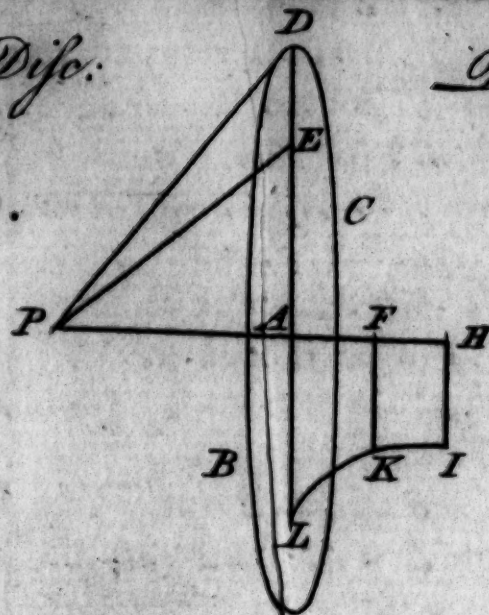


Fig. 13.

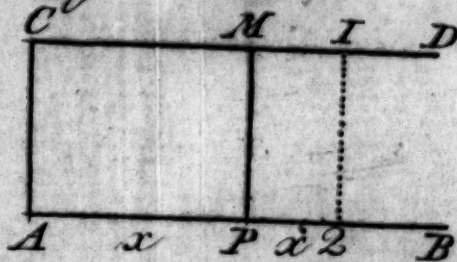


Fig. 14.

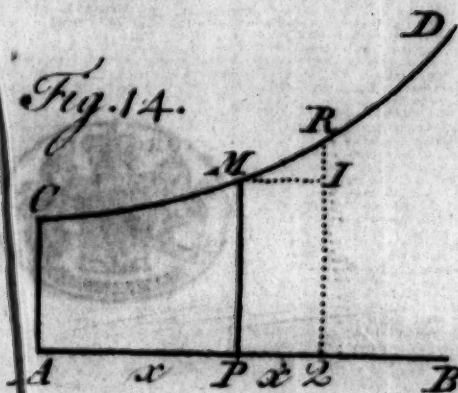


Fig. 15.

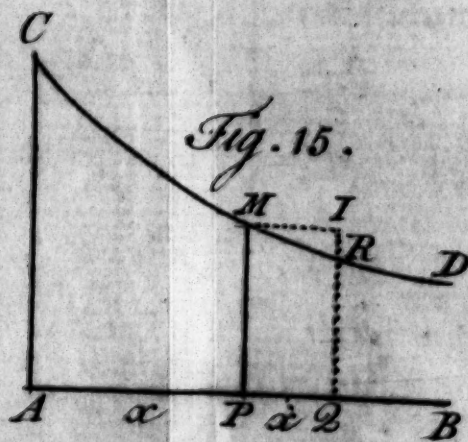


Fig. 16.

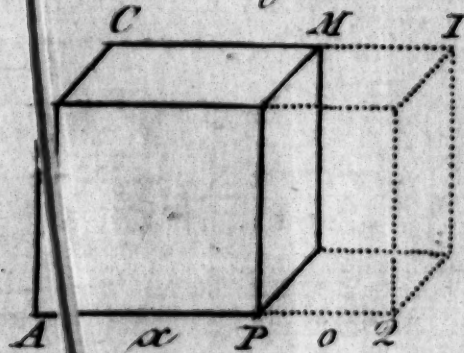


Fig. 11. 12. 13. 14.

